NEW PARAMETERIZATION
OF THE RESONANT PRODUCTION AMPLITUDES
NEAR AN INELASTIC THRESHOLD

Motivation:

1. better understanding of meson-meson interactions and hadron spectroscopy;
2. more precise determination of parameters corresponding to resonances situated close to an inelastic threshold;
3. demonstration of a limited applicability of the Flatté formula, commonly used in experimental analyses (S.M. Flatté, Phys. Lett. 63B (1976) 224);
4. development of a new unitary parameterization satisfying a generalized Watson's theorem of final state interactions near an inelastic threshold (a short description is given in arXiv:0804.3479 [hep-ph]).

Flatté-parameterization of the production amplitudes

$$A_i \sim \frac{M_R\sqrt{\Gamma_0\Gamma_i}}{M_R^2 - E^2 - iM_R(\Gamma_1+\Gamma_2)}$$

$i = 1$, channel $\pi\eta$; $i = 2$, channel $K\overline{K}$, S-wave,

E = effective mass (c.m. energy), M_R = resonance mass,

$\Gamma_1 = g_1 k_1$, $\quad k_1 = \frac{1}{2E}\sqrt{[E^2-(m_\pi+m_\eta)^2][E^2-(m_\pi-m_\eta)^2]}$,
$k_1 = \eta$ c.m. momentum,
$E_0 = 2m_K$ (the threshold energy), $q = k_1(E_0)$, $\Gamma_0 = g_1 q$.

Above the threshold $\Gamma_2 = g_2 k_2$, $k_2 = \sqrt{\frac{E^2}{4} - m_K^2}$, k_2 = kaon c.m. momentum,

below the threshold $\Gamma_2 = g_2 i p_2$, $p_2 = \sqrt{m_K^2 - \frac{E^2}{4}}$.

3 parameters: M_R, g1 and g2 - coupling constants.

New formula for the production amplitudes

$$A_i \sim \frac{1}{W(E)}$$

$$W(E) = M_R^2 - E^2 - iM_R g_1 q - iM_R g_2 k_2 + N\, k_2^2$$

N - a new complex constant.

$E^2 = E_0^2 + 4k_2^2\,, \quad E_0 = 2m_K,$

$$\frac{W(E)}{M_R g_2} = \frac{1}{A} - ik_2 + \frac{1}{2} R k_2^2$$

$Re(\frac{1}{A}) = \frac{M_R^2 - E_0^2}{M_R g_2}, \quad Im(\frac{1}{A}) = -\frac{g_1}{g_2}\, q\,, \quad A$ =complex scattering length.

Complex effective range $R = \frac{2N-8}{M_R g_2}$ is equivalent to $N = \frac{1}{2} M_R g_2 R + 4.$

Flatté approximation: $N = 0 \;\rightarrow ReR = \frac{-8}{M_R g_2}, \quad ImR = 0$.

Elastic scattering amplitude in the second channel

Without a coupling to the first channel

$T_{22} = \frac{1}{k_2 \cot \delta_2 - ik_2} \equiv \frac{\sin \delta_2}{k_2} e^{i\delta_2}$.

Near threshold expansion: $k_2 \cot \delta_2 = \frac{1}{a} + \frac{1}{2} r k_2^2,$

δ_2 - phase shift, a - real scattering length, r - real effective range.

In presence of a coupling to the first channel

$T_{22} = \frac{1}{2ik_2}(\eta e^{2i\delta_2} - 1),$ η - inelasticity.

Effective range expansion near the $K\bar{K}$ threshold:

$$T_{22} = \frac{1}{\frac{1}{A} - i\ k_2 + \frac{1}{2}\ R\ k_2^2}$$

A - complex scattering length, R - complex effective range .

Elastic scattering amplitude in the first channel

$$T_{11} = \frac{1}{2ik_1}(\eta e^{2i\delta_1} - 1)\ .$$

At the $K\overline{K}$ threshold $\eta = 1$, $\delta_1(q) \equiv \delta_0$ and $\quad T_{11}(0) = \frac{\sin\delta_0}{q} e^{i\delta_0}$.

New formula for T_{11} above the $K\overline{K}$ threshold:

$$T_{11} = \frac{e^{i\delta_0}}{k_1} \frac{\sin\delta_0 + i\ Im\ (e^{-i\delta_0}\ A)\ k_2 - \frac{1}{2}\ Im\ (e^{-i\delta_0}\ A\ R)\ k_2^2}{1 - i\ A\ k_2 + \frac{1}{2}\ A\ R\ k_2^2}.$$

5 independent parameters: Re A, Im A, Re R, Im R and δ_0.

Below the $K\overline{K}$ threshold $k_2 \to ip_2$ (analyticity of T_{11}).

Flatté limit: $\delta_0 = \phi_A$, where ϕ_A= phase of A, Im R=0 .

Poles of the scattering amplitudes

All the amplitudes have a common denominator: $T_{ij} \sim D(k_2)^{-1}, \quad i, j = 1, 2,$

$$D(k_2) = 1 - \; i\,A\,k_2 \; + \; \tfrac{1}{2}\,A\,R\,k_2^2\,.$$

Two amplitude poles at z_1 and z_2 are zeroes of $D(k_2)$ in the complex k_2 plane :

$$z_{1,2} = \tfrac{i}{R} \pm \sqrt{-\tfrac{1}{R^2} - \tfrac{2}{AR}}\,.$$

Relations to the scattering length A and to the effective range R:

$$A = -i(\tfrac{1}{z_1} + \tfrac{1}{z_2}), \qquad\qquad R = \tfrac{2i}{z_1+z_2}.$$

Flatté approximation: $ImR = 0 \;\rightarrow\; Re\,z_1 \;= -Re\,z_2$. This constraint has an important impact on the values of the complex energy poles

$$E_{1,2} = \sqrt{E_0^2 + 4z_{1,2}^2}.$$

Unitarity relations

$$S^\dagger S = 1, \quad \text{S-matrix.}$$

Relation to T-matrix: $S_{ij} = \delta_{ij} + 2\,i\,\sqrt{k_i k_j}\,T_{ij}, \quad i, j = 1, 2.$

Below the threshold ($E \leq E_0$):

$$Im\,T_{11} = k_1 |T_{11}|^2.$$

Above the threshold ($E \geq E_0$):

$$\begin{aligned} Im\,T_{11} &= k_1\,|T_{11}|^2 + k_2\,|T_{12}|^2 \\ Im\,T_{22} &= k_2\,|T_{22}|^2 + k_1\,|T_{12}|^2 \\ Im\,T_{12} &= k_1\,T_{11}\,T_{12}^* + k_2\,T_{12}\,T_{22}^*. \end{aligned}$$

In a matrix notation: $Im\,T = T^\dagger\,k\,T$

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \qquad k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

Watson's theorem and its generalization above the inelastic threshold

Assumption: no strong interactions in the initial state.

Watson's theorem: below the inelastic threshold

$$Im\, A_1 = k_1\, T_{11}\, A_1^*, \qquad A_1 \text{ - production amplitude;}$$

phase of A_1 = phase of $T_{11} = \delta_1$.

Generalization to two coupled channels:

$$Im\, A_1 = k_1\, T_{11}\, A_1^* + k_2\, T_{12}\, A_2^*$$
$$Im\, A_2 = k_2\, T_{22}\, A_2^* + k_1\, T_{21}\, A_1^*.$$

In a matrix notation:

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

$Im\, A = T\, k\, A^*$ or equivalently $A = S\, A^*$.

Parameterization of the production amplitudes

$$A_1 = f_1\, T_{11} \;+\; f_2\, T_{12}$$
$$A_2 = f_1\, T_{12} \;+\; f_2\, T_{22}$$

If there are no strong initial state interactions f_1, f_2 are real functions of energy (or momentum k_2). Then the generalized Watson's theorem is satisfied.
The two-channel scattering amplitudes in a new approach:
$T_{ij}(k_2) = \frac{N_{ij}(k_2)}{D(k_2)}$, N_{ij} -numerators defined previously.

Then

$$A_1 = \frac{B_1(k_2)}{D((k_2)}, \quad A_2 = \frac{B_2(k_2)}{D(k_2)},$$

$$B_1 = f_1(k_2)N_{11}(k_2) + f_2(k_2)N_{12}(k_2)$$
$$B_2 = f_1(k_2)N_{12}(k_2) + f_2(k_2)N_{22}(k_2).$$

Possible approximation of $f_i(k_2)$ near an inelastic threshold:
$f_1(k_2) = n_1 + \beta_1 k_2^2, \quad f_2(k_2) = n_2 + \beta_2 k_2^2,$
n_1, n_2 - normalization constants, $n_1, \beta_1, n_2, \beta_2$ - real .

Numerical example: a case of the $a_0(980)$ resonance

Two coupled channels: 1. $\pi\eta$, 2. $K\bar{K}$, S-wave, isospin 1.

A coupled channel formalism for the separable meson-meson interactions in two or three channels: L. L., *Meson spectroscopy and separable potentials*, Acta Physica Polonica B27 (1996) 1835.

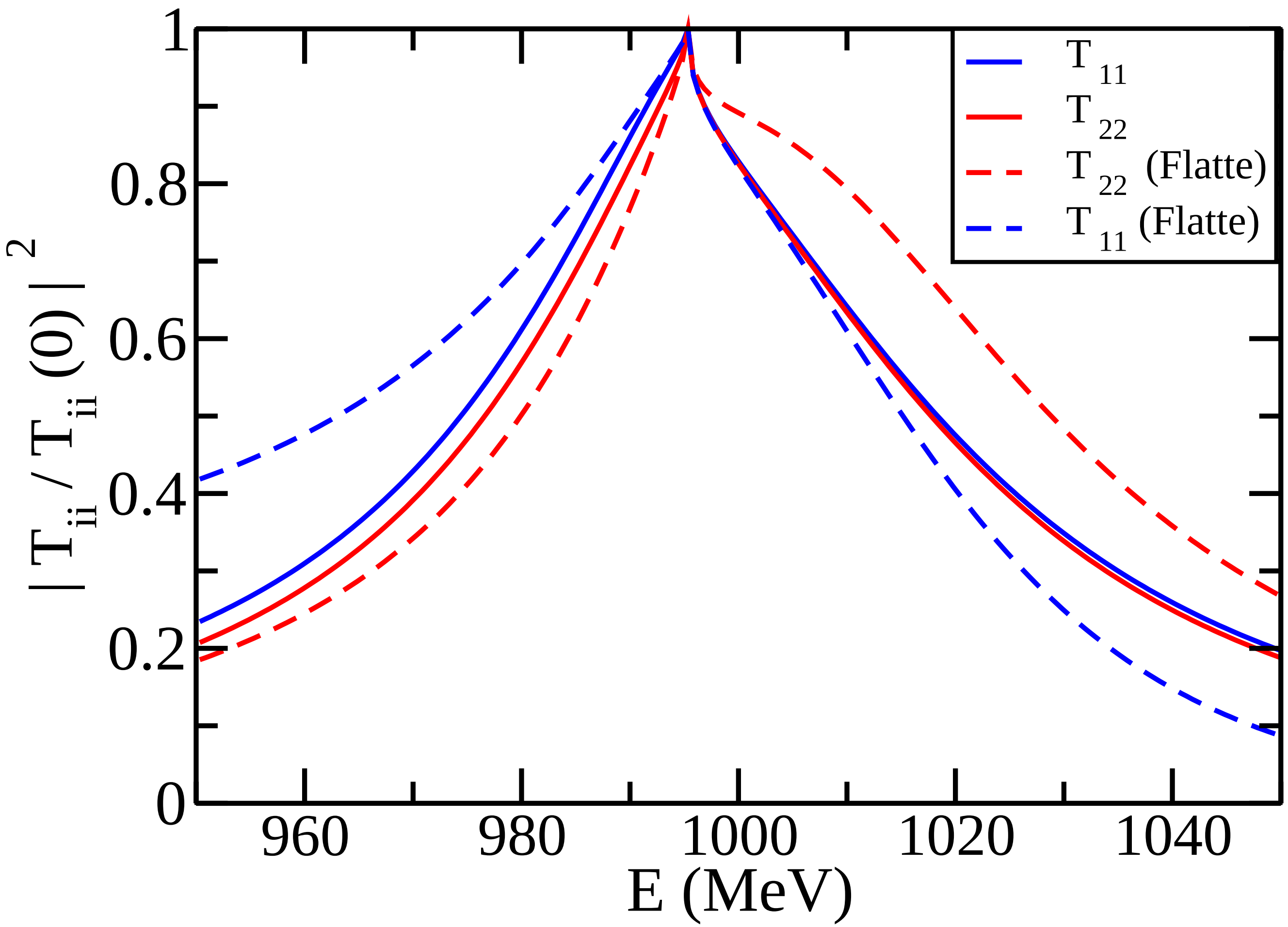
Application of this formalism to study the $\pi\eta$ and the $K\bar{K}$ interactions by Agnieszka Furman and L. L. , in Physics Letters B538 (2002) 266.

The parameters of this model were fixed using the data of the Crystal Barrel and of the E-852 Collaborations.
The following threshold parameters can be calculated:

$Re\ A = 0.17$ fm, $Im\ A = 0.41$ fm, $Re\ R = -11.32$ fm, $Im\ R = -3.18$ fm.

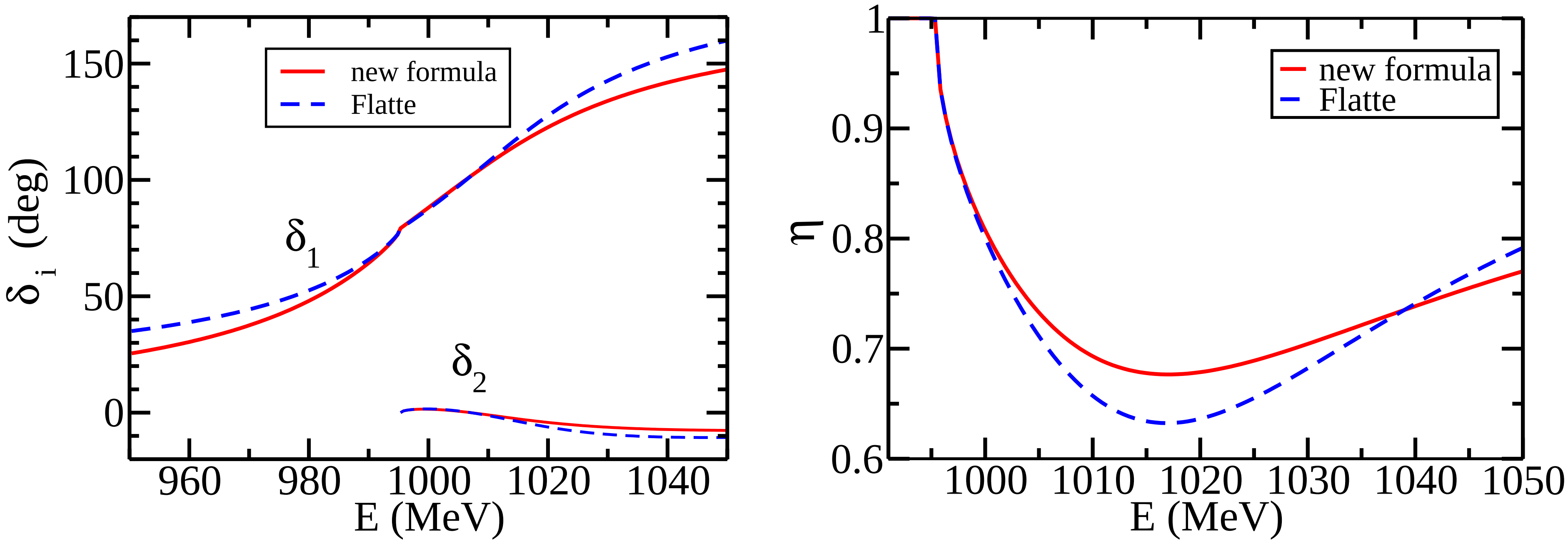
The imaginary part of the effective range cannot be neglected: $Im\ R \neq 0$!

Moduli of the scattering amplitudes

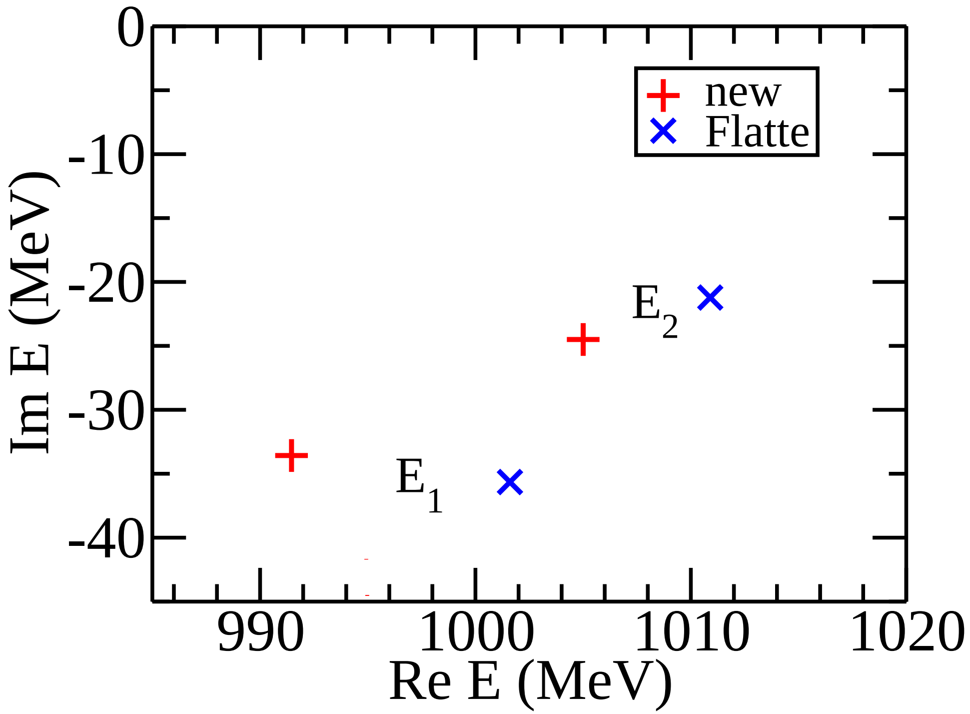


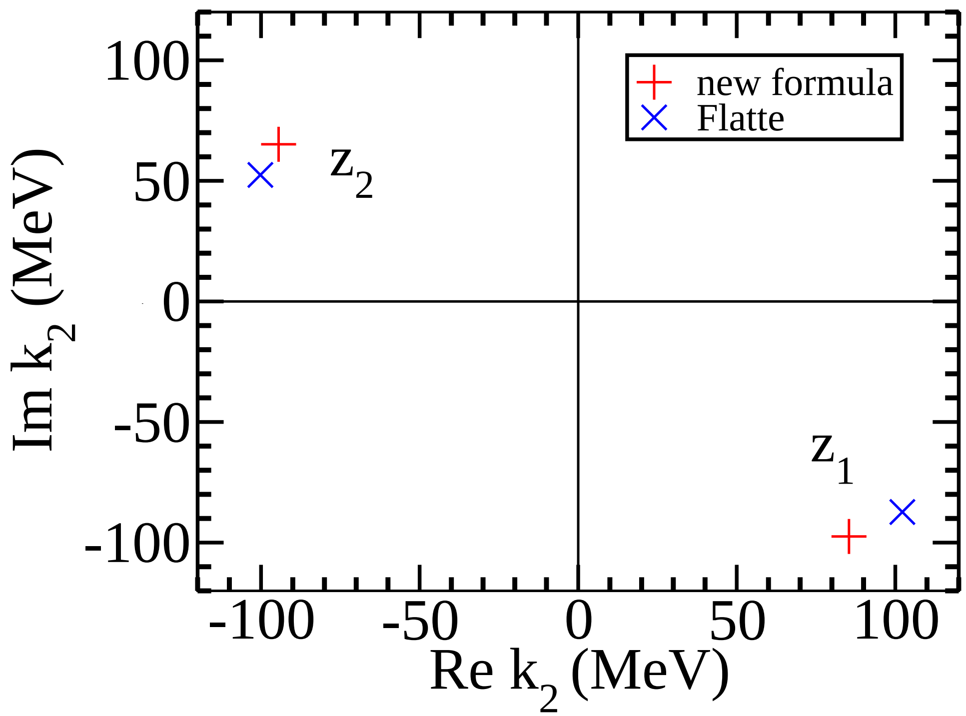
Remark: substantial deviations from the Flatté formula!

Phase shifts and inelasticity



Position of poles





Remark: a shift of Re E_1 by more than 10 MeV.

Summary: replace the Flatté production amplitudes A_i by

$$A_1 = \frac{B_1}{D}, \quad A_2 = \frac{B_2}{D}, \qquad D = 1 - \; i\, A\, k_2 \; + \; \tfrac{1}{2}\, A\, R\, k_2^2$$

$$B_1 = f_1 N_{11} + f_2 N_{12}, \qquad B_2 = f_1 N_{12} + f_2 N_{22}$$

$$N_{11} = \frac{e^{i\delta_0}}{k_1}[\; \sin\delta_0 \; + \; i\, Im\,(e^{-i\delta_0}\, A)\, k_2 \; - \; \tfrac{1}{2}\, Im\,(e^{-i\delta_0}\, A\, R)\, k_2^2 \;]$$

$$N_{12} = \frac{1}{\sqrt{k_1}}\, e^{i\delta_0}\, \sqrt{Im\, A \; - \; \tfrac{1}{2}\, |A|^2\, Im\, R\, k_2^2}, \qquad N_{22} = A$$

$$f_1 = n_1 + \beta_1 k_2^2, \quad f_2 = n_2 + \beta_2 k_2^2.$$

Channel momenta: $k_1 = \frac{1}{2E}\sqrt{[E^2 - (m_\pi + m_\eta)^2][E^2 - (m_\pi - m_\eta)^2]}$,

above the threshold $k_2 = \sqrt{\frac{E^2}{4} - m_K^2}$, below: $k_2 \to i p_2, \;\; p_2 = \sqrt{m_K^2 - \frac{E^2}{4}}$.

Generalization to a case of unequal masses $m_a \neq m_b$ in the second channel:

$$k_2 = \frac{1}{2E}\sqrt{[E^2 - (m_a + m_b)^2][E^2 - (m_a - m_b)^2]}.$$

Parameters to be fitted from experiments: complex A,R, real $\delta_0, n_1, n_2, \beta_1, \beta_2$.

CONCLUSIONS

1. The Flatté formula is not sufficiently accurate in analyses of new data on the resonance production near inelastic thresholds. Its application can lead to a substantial displacement of the resonance pole positions.

2. A simple unitary parameterization, satisfying a generalized Watson theorem for the production amplitudes, is proposed. It contains more measurable parameters than those included in the Flatté formula.
   A knowledge of a complex scattering length and a complex effective range is necessary in the description of meson-meson interactions near inelastic thresholds. A near threshold resonance is characterized by two distinct complex energy poles and not just by one mass parameter M_R.

3. The new formulae can be applied in numerous analyses of present and future experiments (for example Belle, BaBar, CLEO, BES, KLOE, COSY, Tevatron, LHCb, HERA, JLab, PANDA ...) and also to reanalyse older experiments in order to improve our information about the hadron spectroscopy and the reaction mechanism.

Transition amplitude from the first to the second channel

$$T_{12} = \frac{1}{2\sqrt{k_1 k_2}} \sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)}.$$

New formula for T_{12} near the threshold:

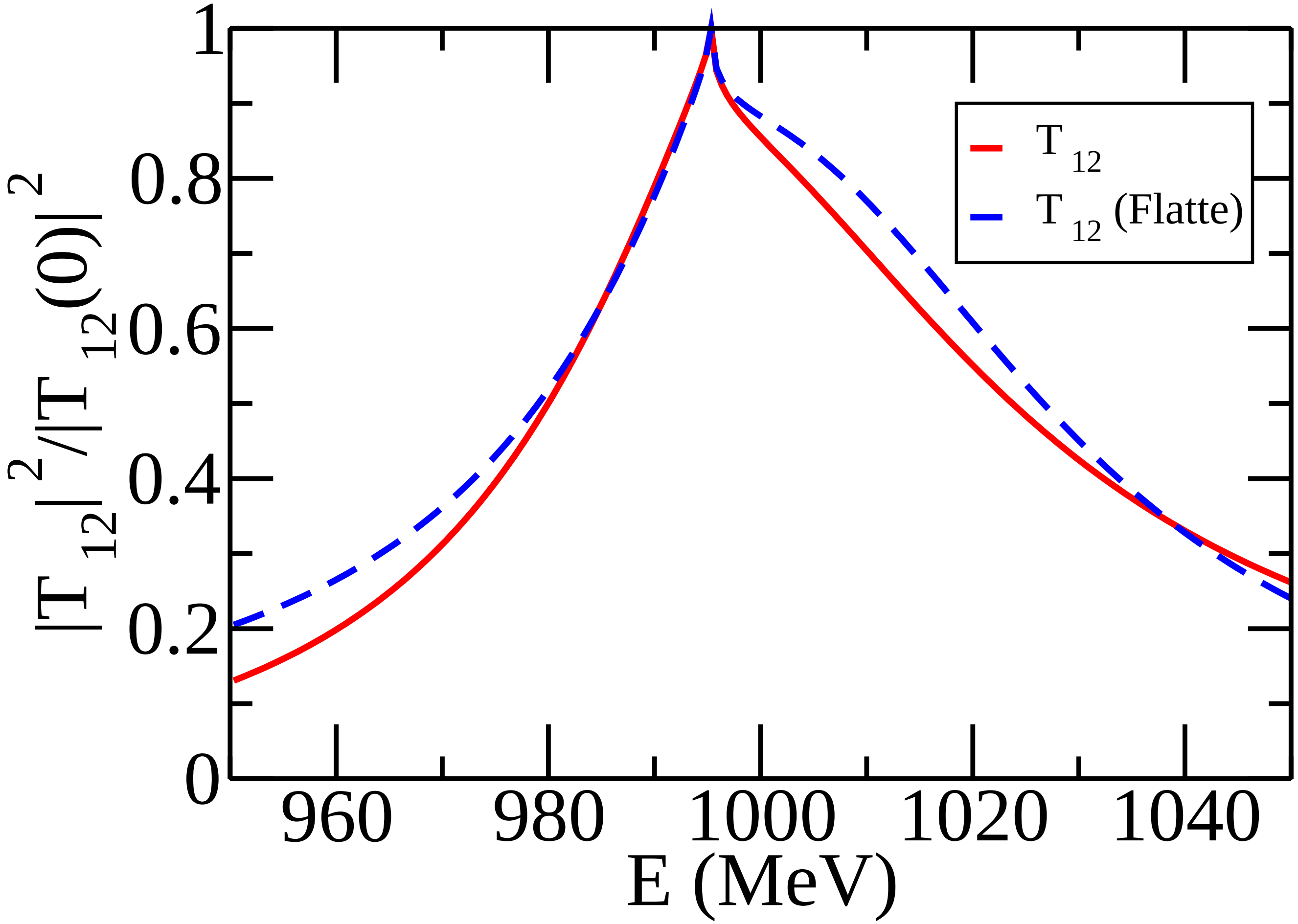
$$T_{12} = \frac{1}{\sqrt{k_1}}\, e^{i\delta_0} \frac{\sqrt{Im\ A - \frac{1}{2}\ |A|^2\ Im\ R\ k_2^2}}{1 - \ i\ A\ k_2 + \frac{1}{2}\ A\ R\ k_2^2}.$$

Remark: if $ImA = ImR = 0$ then $T_{12} = 0$ (no transition between channels).

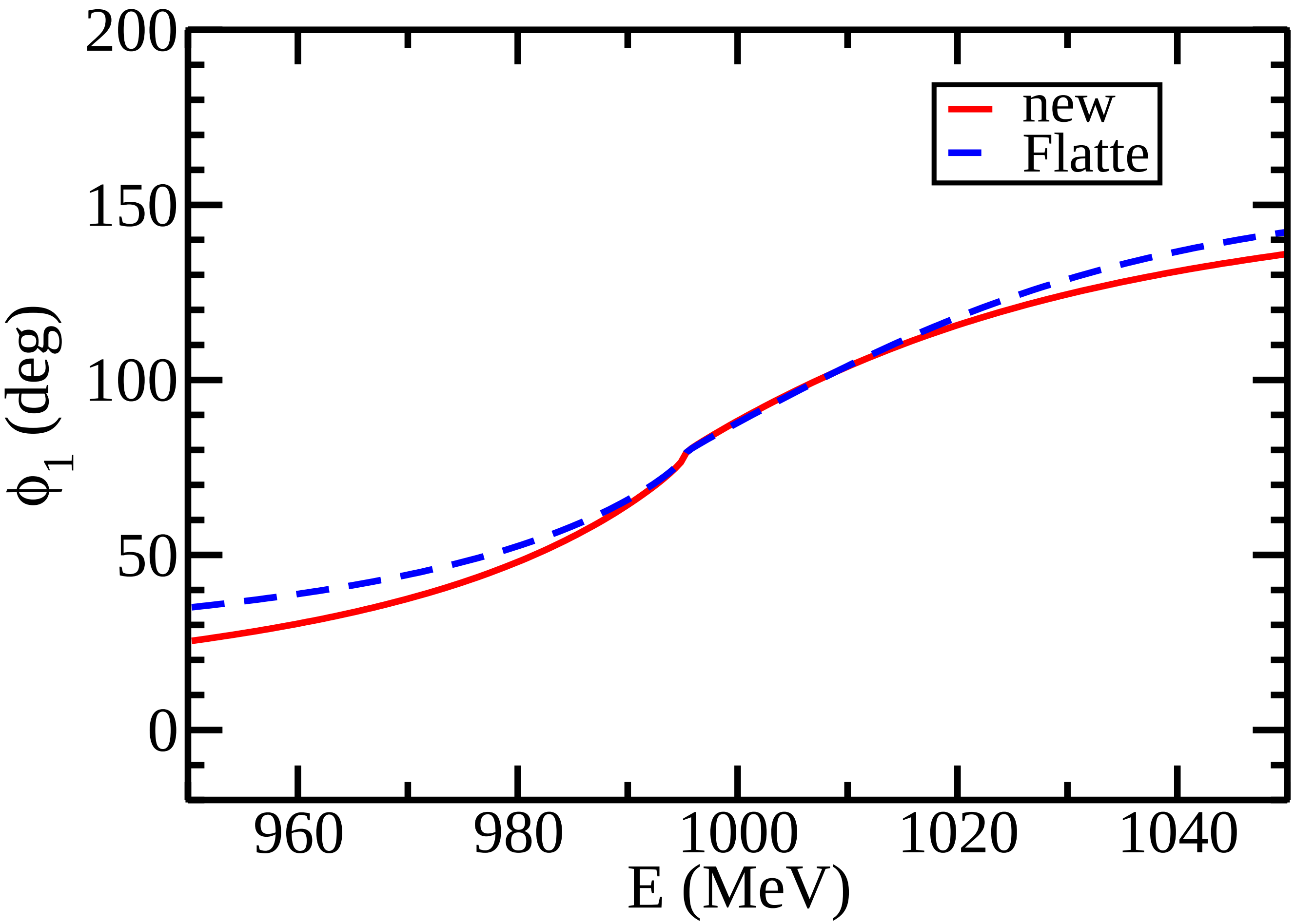
Flatté limit:

$$T_{12}^F = \frac{M_R \sqrt{g_1\ g_2}}{W^F(E,N=0)}.$$

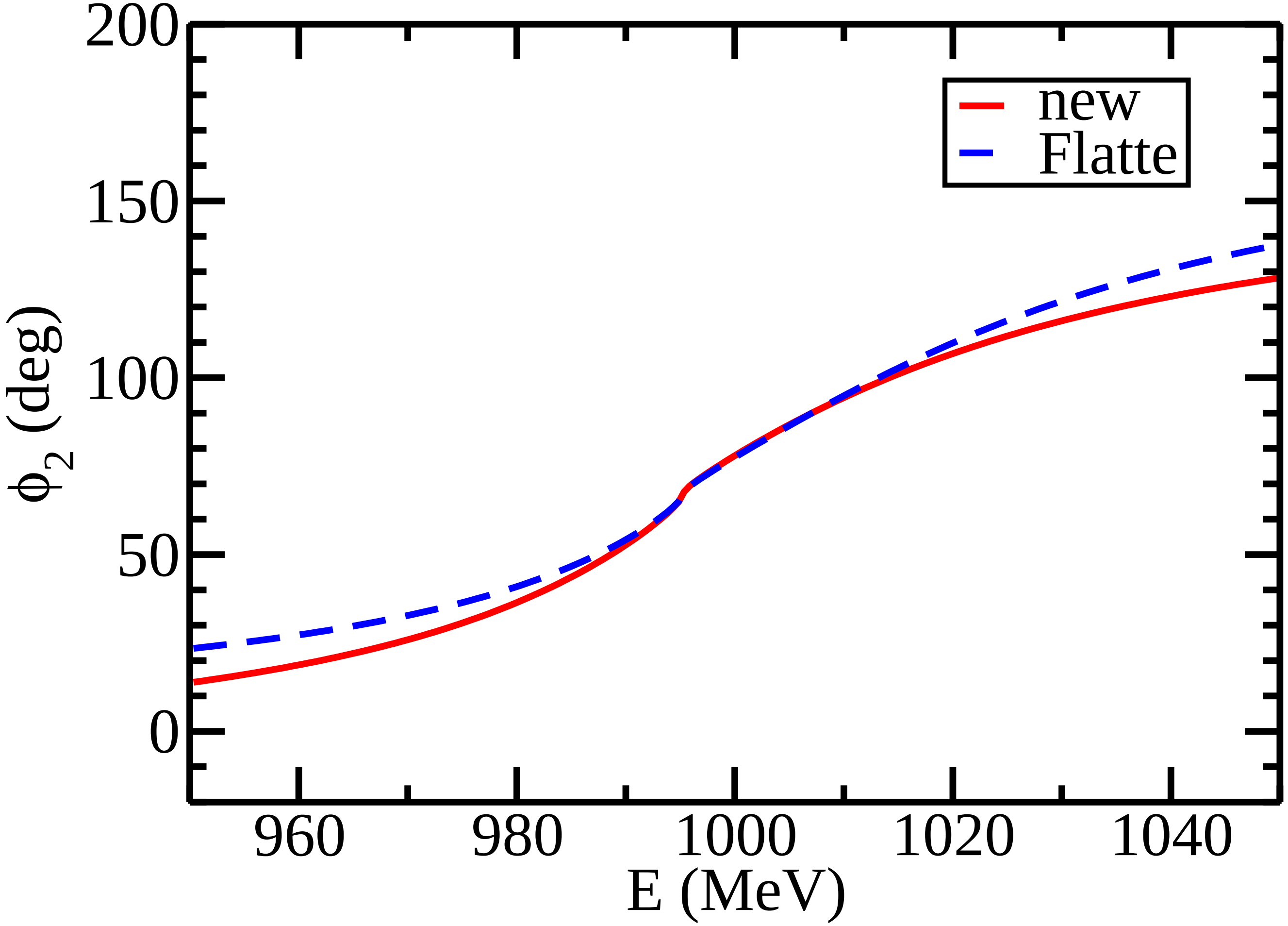
Modulus of the transition amplitude T_{12}



Phase of the T_{11} amplitude



Phase of the T_{22} amplitude



Near threshold expansion of phase shifts and inelasticity

Below the threshold: $E^2 = E_0^2 - 4\,|k_2|^2,\; k_2 = i\,|k_2|,$

$$\delta_1 \approx \delta_0 - Im\,A\,|k_2| + \tfrac{1}{2} Im\,(A^2 + AR)\,|k_2|^2 + 0(k_2^3),$$

$$\eta = 1.$$

Above the threshold: $E^2 = E_0^2 + 4\,k_2^2,$

$$\delta_1 \approx \delta_0 - \tfrac{1}{2} Im\,(A^2 + AR)\,k_2^2 + 0(k_2^4),$$

$$\delta_2 \approx Re\,A\,k_2 + 0(k_2^3),$$

$$\eta \approx 1 - 2\,Im\,A\,k_2 + 2\,(ImA)^2\,k_2^2 + 0(k_2^3).$$

Unitarity: $\eta \leq 1 \rightarrow Im\,A \geq 0.$