Spectroscopic implications from the analysis of processes $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta, \eta\eta'$

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- **Analysis of the isoscalar-tensor sector**
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Motivation

We present results of the coupled-channel analysis of data on processes $\pi\pi \longrightarrow \pi\pi, K\overline{K}, \eta\eta, \eta\eta'$ in the channels with $I^G J^{PC} = 0^+0^{++}$ and 0^+2^{++} and on the $\pi\pi$ scattering in the channel with 1^+1^{--}. The scalar sector is problematic up to now especially due to an assignment of the discovered mesonic states to quark-model configurations in spite of a big amount of work devoted to these problems (see, e.g., *V.V.Anisovich, IJMP A **21**, 3615 (2006)* **and references therein). An exceptional interest to this sector is supported by the fact that there, possibly indeed, we deal with a glueball $f_0(1500)$ (see, e.g.,** *C.Amsler, F.E.Close, PR D **53**, 295 (1996); W.-M. Yao* et al. *(PDG), J.Phys. G **33** (2006) 1.***).**

The investigation of vector mesons is up-to-date subject due to their role in forming the electromagnetic structure of particles and because our knowledge about these mesons is still too incomplete (e.g., in the PDG issue the masses of $\rho(1450)$ is ranging from 1250 to 1582 MeV, and it is for$\rho(1450)$ a quoted averaged total width of only 147 MeV!)

The $\pi\pi$ interaction plays a central role in physics of strongly interacting particles and, therefore, it has always been an object of continuous investigation: *I. Caprini, G. Colangelo, and H. Leutwyler, IJMP A **21** (2006) 954; R. Kaminski, J.R. Pelaez, and F.J. Yndurain, PR D **74** (2006) 014001; Erratum-ibid. D **74** (2006) 079903; A.A. Osipov, A.E. Radzhabov, and M.K. Volkov, arXiv: hep-ph/0603130; V. Bernard, A.A. Osipov, and U.G. Meissner, PL B**285** (1992) 119.*

In the tensor sector, the nine states ($f_2(1430)$, $f_2(1565)$, $f_2(1640)$, $f_2(1810)$, $f_2(1910)$, $f_2(2000)$, $f_2(2020)$, $f_2(2150)$, $f_2(2220)$) must be confirmed in various experiments and analyses.

In the analysis of $p\bar{p} \to \pi\pi, \eta\eta, \eta\eta'$, five resonances – $f_2(1920)$, $f_2(2000)$, $f_2(2020)$, $f_2(2240)$ and $f_2(2300)$ – have been obtained, one of which ($f_2(2000)$) is a candidate for the glueball (*V.V.Anisovich et al., IJMP A **20, 6327 (2005)*).**

Method of analysis

We analysed experimental data using both *a model-independent method* **based only on the analyticity and unitarity (***D.Krupa, V.A.Meshcheryakov, Yu.S.Surovtsev, NC A* ***109****, 281 (1996)***) and** *multichannel Breit–Wigner forms***. In both methods, we parametrized the S-matrix elements $S_{\alpha\beta}$ ($\alpha,\beta = 1(\pi\pi), 2(K\overline{K}), 3(\eta\eta$ or $\eta\eta'))$ using the Le Couteur-Newton relations (***K.J.LeCouteur, Proc.Roy.Soc. A* ***256****, 115 (1960); R.G.Newton, J.Math.Phys.* ***2****, 188 (1961); M.Kato, Ann.Phys.* ***31****, 130 (1965)***). They express the S-matrix elements of all coupled processes in terms of the Jost matrix determinant $d(k_1, \cdots, k_n)$ that is a real analytic function with the only square-root branch-points at channel momenta $k_\alpha = 0$.**

Our model-independent method which essentially utilizes an uniformizing variable can be used only for the 2-channel case and under some conditions for the 3-channel one. Only in these cases we obtain a simple symmetric (easily interpreted) picture of the resonance poles and zeros of the *S*-matrix on an uniformization plane. The important branch points, corresponding to the thresholds of the coupled channels and to the crossing ones, are taken into account in the uniformizing variable.

A resonance is represented by three and seven types pair of complex-conjugate clusters (of poles and zeros on the Riemann surface) in the 2- and 3-channel cases, respectively (*D.Krupa et al., NC A* ***109****, 281 (1996)***). The cluster kind is related to the nature of state.**

Analysis of the isoscalar-scalar sector

Considering the S-waves of processes

$$\pi\pi \to \pi\pi, K\overline{K}, \eta\eta, \eta\eta'$$

in the model-independent approach, we performed 2 variants of the 3-channel analysis.
Variant I: A combined analysis of $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta$.
Variant II: Analysis of $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta'$.
Influence of the $\eta\eta'$-channel in the I case and of $\eta\eta$ in the II one are taken into account in the background.
Here the left-hand cuts are neglected in the Riemann-surface structure, and contributions on these cuts are taken into account also in the background.

Under neglecting the $\pi\pi$-threshold branch point (however, unitarity on the $\pi\pi$-cut is taken into account), the uniformizing variable is:

$$w = \frac{k_2 + k_3}{\sqrt{m_\eta^2 - m_K^2}} \qquad \text{for variant I,}$$

and

$$w' = \frac{k_2' + k_3'}{\sqrt{\frac{1}{4}(m_\eta + m_{\eta'})^2 - m_K^2}} \qquad \text{for variant II.}$$

All, related to variant II, is marked by prime.

On the w-plane, the Le Couteur-Newton relations are

$$S_{11} = \frac{d^*(-w^*)}{d(w)}, \qquad S_{22} = \frac{d(-w^{-1})}{d(w)}, \qquad S_{33} = \frac{d(w^{-1})}{d(w)},$$

$$S_{11}S_{22} - S_{12}^2 = \frac{d^*(w^{*-1})}{d(w)}, \qquad S_{11}S_{33} - S_{13}^2 = \frac{d^*(-w^{*-1})}{d(w)}.$$

$$d = d_B d_{res}, \qquad d_{res}(w) = w^{-\frac{M}{2}} \prod_{r=1}^{M} (w + w_r^*)$$

M is the number of resonance zeros.

$$d_B = \exp[-i \sum_{n=1}^{3} \frac{k_n}{m_n}(\alpha_n + i\beta_n)],$$

$$\alpha_n = a_{n1} + a_{n\sigma}\frac{s - s_\sigma}{s_\sigma}\theta(s - s_\sigma) + a_{nv}\frac{s - s_v}{s_v}\theta(s - s_v),$$

$$\beta_n = b_{n1} + b_{n\sigma}\frac{s - s_\sigma}{s_\sigma}\theta(s - s_\sigma) + b_{nv}\frac{s - s_v}{s_v}\theta(s - s_v).$$

s_σ – the $\sigma\sigma$ threshold; s_v – the combined threshold of many opening channels in the range of $\sim$ 1.5 GeV ($\eta\eta',\ \rho\rho,\ \omega\omega$).

In variant II (the uniformizing variable w'),

$$a'_{n\eta}\frac{s - 4m_\eta^2}{4m_\eta^2}\theta(s - 4m_\eta^2) \quad \text{and} \quad b'_{n\eta}\frac{s - 4m_\eta^2}{4m_\eta^2}\theta(s - 4m_\eta^2)$$

should be added to α'_n and β'_n.

The $\pi\pi$ scattering data from the threshold to 1.89 GeV are taken from: *B.Hyams* et al., *NP B* ***64****, 134 (1973); ibid.* ***100****, 205 (1975); A.Zylbersztejn* et al., *PL B* ***38****, 457 (1972); P.Sonderegger, P.Bonamy, in Proc. 5th Intern. Conf. on Elem. Part., Lund, 1969, paper 372; J.R.Bensinger* et al., *PL B* ***36****, 134 (1971); J.P.Baton* et al., *PL B* ***33****, 525, 528 (1970); P.Baillon* et al., *PL B* ***38****, 555 (1972); L.Rosselet* et al., *PR D* ***15****, 574 (1977); A.A.Kartamyshev* et al., *Pis'ma v ZhETF* ***25****, 68 (1977); A.A. Bel'kov* et al., *Pis'ma v ZhETF* ***29****, 652 (1979)*. **For $\pi\pi \to K\overline{K}$, practically all the accessible data are used:** *W.Wetzel* et al., *NP B* ***115****, 208 (1976); V.A.Polychronakos* et al., *PR D* ***19****, 1317 (1979); P.Estabrooks, PR D* ***19****, 2678 (1979); D.Cohen* et al., *PR D* ***22****, 2595 (1980); G.Costa* et al., *NP B* ***175****, 402 (1980); A.Etkin* et al., *PR D* ***25****, 1786 (1982)*.

For $\pi\pi \to \eta\eta$, we used data for $|S_{13}|^2$ from the threshold to 1.72 GeV (*F.Binon* et al., *NC A* ***78****, 313 (1983)***).**

For $\pi\pi \to \eta\eta'$, the data for $|S_{13}|^2$ from the threshold to 1.813 GeV are taken from (*F. Binon* et al., *NC A* ***80****, 363 (1984)***).**

We included all 5 resonances discussed below 1.9 GeV.

In <u>variant I</u>, we got satisfactory description: for the $\pi\pi$ scattering, $\chi^2/\mathrm{NDF} \approx 1.35$; for $\pi\pi \to K\overline{K}$, $\chi^2/\mathrm{NDF} \approx 1.77$; for $\pi\pi \to \eta\eta$, $\chi^2/\mathrm{N.exp.points} \approx 0.86$. The total χ^2/NDF is $345.603/(301-40) \approx 1.32$.

The background parameters are: $a_{11} = 0.2006$, $a_{1\sigma} = 0.0146$, $a_{1v} = 0$, $b_{11} = 0$, $b_{1\sigma} = -0.01025$, $b_{1v} = 0.0542$, $a_{21} = -0.6986$, $a_{2\sigma} = -1.4207$, $a_{2v} = -5.958$, $b_{21} = 0.047$, $b_{2\sigma} = 0$, $b_{2v} = 6.888$, $b_{31} = 0.6511$, $b_{3\sigma} = 0.3404$, $b_{3v} = 0$; $s_\sigma = 1.638\ \mathrm{GeV}^2$, $s_v = 2.084\ \mathrm{GeV}^2$.

In <u>variant II</u>, we got description: For the $\pi\pi$ scattering $\chi^2/\mathrm{NDF} \approx 1.0$! for $\pi\pi \to K\overline{K}$ $\chi^2/\mathrm{NDF} \approx 1.62$; for $\pi\pi \to \eta\eta'$ $\chi^2/\mathrm{N.exp.points} \approx 0.36$. The total χ^2/NDF is $282.682/(293-38) \approx 1.11$!

The background parameters are: $a'_{11} = 0.0111$, $a'_{1\eta} = -0.058$, $a'_{1\sigma} = 0$, $a'_{1v} = 0.0954$, $b'_{11} = b'_{1\eta} = b'_{1\sigma} = 0$, $b'_{1v} = 0.047$, $a'_{21} = -3.439$, $a'_{2\eta} = -0.4851$, $a'_{2\sigma} = 1.7622$, $a'_{2v} = -5.158$, $b'_{21} = 0$, $b'_{2\eta} = -0.7524$, $b'_{2\sigma} = 2.6658$, $b'_{2v} = 1.836$, $b'_{31} = 0.5545$, $s_\sigma = 1.638\ \mathrm{GeV}^2$, $s_v = 2.126\ \mathrm{GeV}^2$.

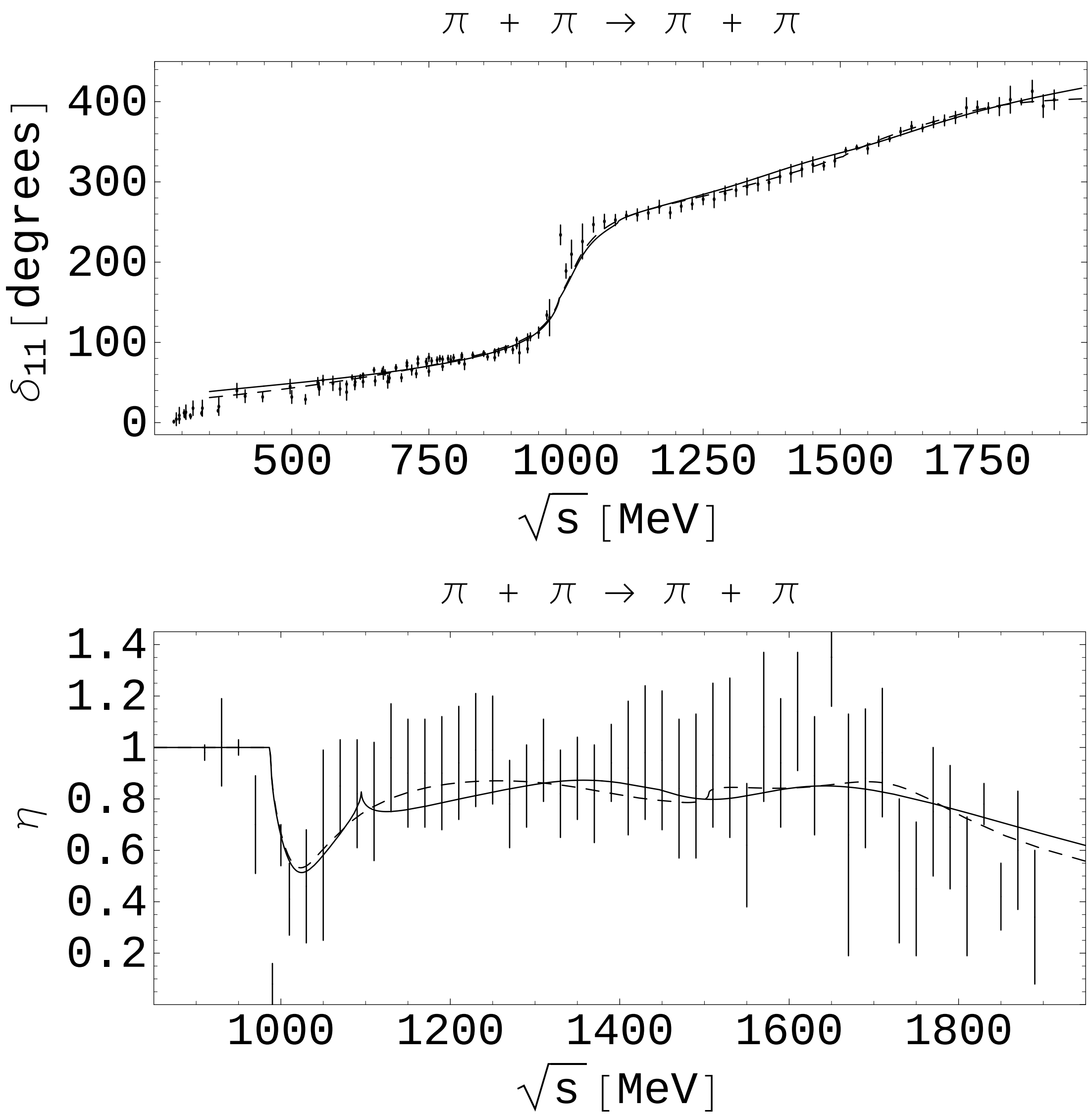


Figure 1: The phase shift and module of the $\pi\pi$-scattering S-wave matrix element. The solid curve – variant I; the dashed curve – variant II.

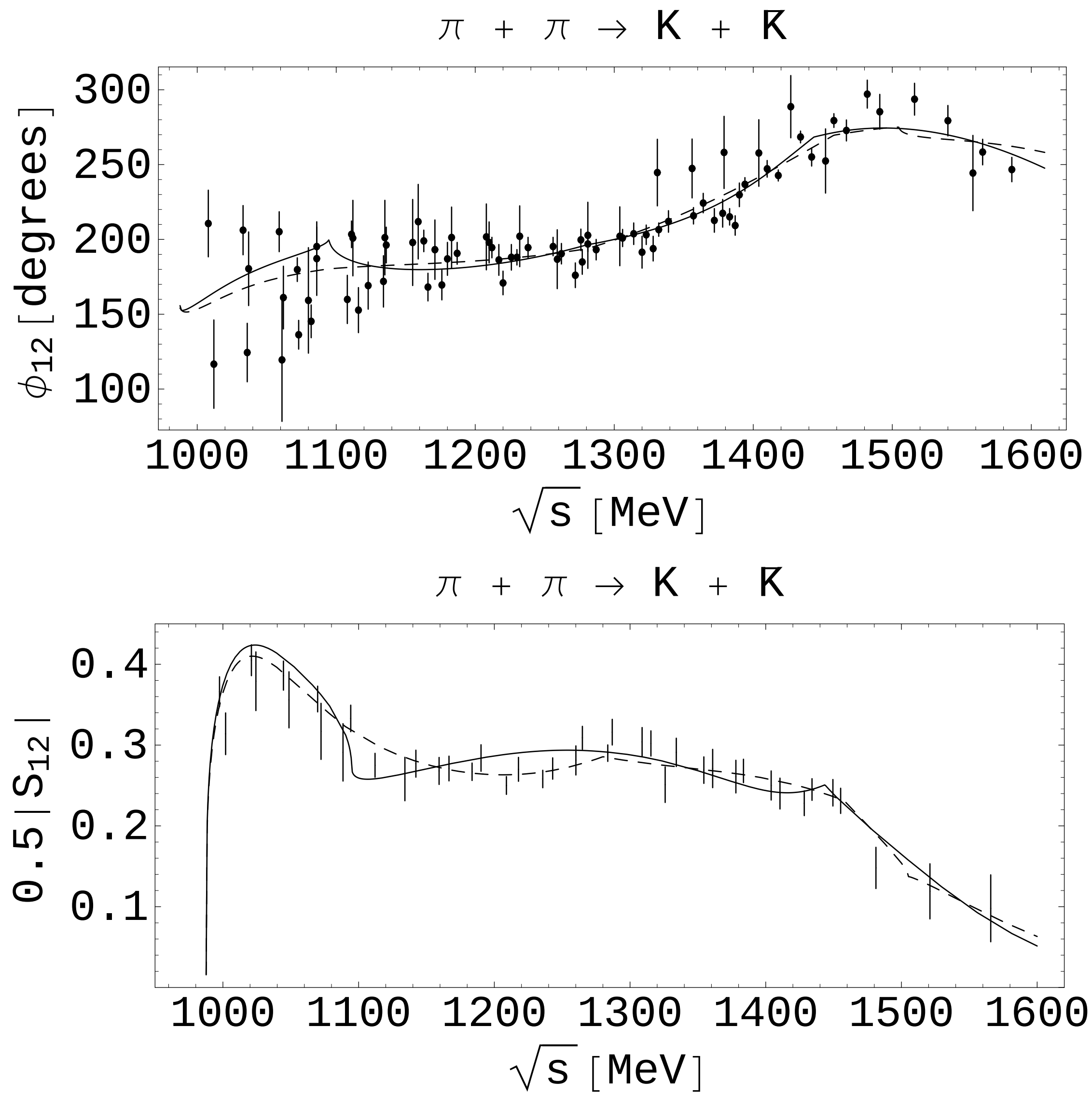


Figure 2: The phase shift and module of the $\pi\pi \to K\overline{K}$ S-wave matrix element. The solid curve – variant I; the dashed curve – variant II.

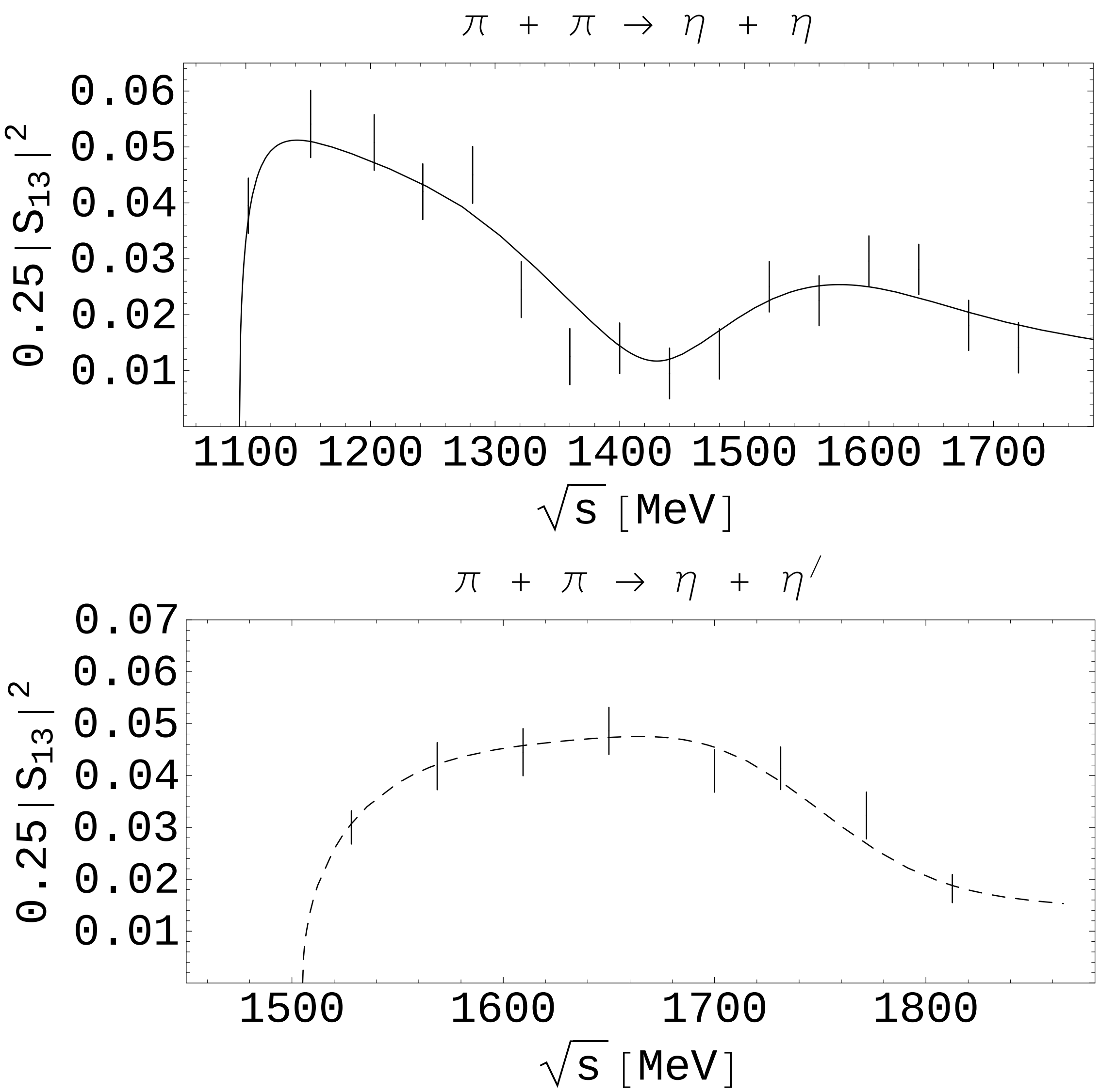


Figure 3: The squared modules of the $\pi\pi \to \eta\eta$ (upper figure) and $\pi\pi \to \eta\eta'$ (lower figure) S-wave matrix elements.

Let us indicate the obtained pole clusters for resonances on 8 sheets of the complex energy plane $\sqrt{s}$, on which the 3-channel S-matrix is determined. $\sqrt{s_r} = \mathrm{E}_r + i\Gamma_r$

Table 1: Pole clusters for the f_0-resonances in variant I.

Sheet		II	III	IV	V	VI	VII	VIII
$f_0(600)$	E_r	598.2±13	585.8±14			505.8±16	518.2±15	
	Γ_r	583±18	583±18			583±18	583±18	
$f_0(980)$	E_r	1013.1±4	983.6±9					
	Γ_r	34.1±6	57.4±10					
$f_0(1370)$	E_r				1398.2±16	1398.2±18	1398.2±18	1398.2±13
	Γ_r				287.4±17	270.6±15	155±9	171.8±7
$f_0(1500)$	E_r	1502.6±11	1479.5±13	1502.6±12	1496.7±12	1498±16	1496.8±12	1502.6±10
	Γ_r	357.1±15	139.4±12	238.7±13	139.9±14	191.2±17	87.36±11	356.5±14
$f_0(1710)$	E_r		1708.2±12	1708.2±10	1708.2±13	1708.2±15		
	Γ_r		142.3±9	160.3±8	323.3±14	305.3±13		

The $f_0(1370)$ and $f_0(1710)$ are represented by the pole clusters corresponding to states with the dominant $s\bar{s}$ component; $f_0(1500)$, with the dominant glueball component.

Table 2: Pole clusters for the f_0-resonances in variant II.

Sheet		II	III	IV	V	VI	VII	VIII
$f_0(600)$	E_r	616.5±8	621.8±10			598.3±11	593±12	
	Γ_r	563±11	563±12			563±14	563±13	
$f_0(980)$	E_r	1009.3±3	986±6					
	Γ_r	32±4	58±5.5					
$f_0(1370)$	E_r		1394.3±9	1394.3±11	1412.7±13	1412.7±14		
	Γ_r		236.3±10	255.7±12	255.7±12	236.3±19		
$f_0(1500)$	E_r	1498.3±11	1502.4±9	1498.3±12	1498.3±13	1494.6±11	1498.3±14	
	Γ_r	198.8±14	236.8±11	193±9	198.8±11	194±8	193±10	
$f_0(1710)$	E_r				1726.1±12	1726.1±13	1726.1±12	1726.1±10
	Γ_r				140.2±9	111.6±8	84.2±8	112.8±7

Note a surprising result obtained for the $f_0(980)$. This state lies slightly above the $K\overline{K}$ threshold and is described by the pole on sheet II and by the shifted pole on sheet III under the $\eta\eta$ threshold without the corresponding poles on sheets VI and VII, as it was expected for standard clusters. This corresponds to the description of the $\eta\eta$ bound state.

Masses and widths of states should be calculated from the pole positions.

$$T^{res} = \sqrt{s}\Gamma_{el}/(m_{res}^2 - s - i\sqrt{s}\Gamma_{tot})$$

Table 3: Masses and total widths of the f_0-resonances (in MeV).

	Variant I		Variant II	
State	m_{res}	Γ_{tot}	m_{res}	Γ_{tot}
$f_0(600)$	835.3	1166	834.9	1126
$f_0(980)$	1013.7	68.2	1009.8	64
$f_0(1370)$	1408.7	343.6	1417.5	511
$f_0(1500)$	1544	714	1511.4	398
$f_0(1710)$	1715.7	321	1729.8	225.6

Analysis of the isovector P-wave of $\pi\pi$ scattering

In this sector we applied both *the model-independent method* **and** *multichannel Breit–Wigner forms***.**

We analyzed data: *S.D. Protopopescu et al., PR D **7**, 1279 (1973); B. Hyams et al., NP B **64**, 134 (1973); P. Estabrooks and A.D. Martin, NP B **79**, 301 (1974)***, for the inelasticity parameter (η) and phase shift of the $\pi\pi$-scattering amplitude (δ) ($S(\pi\pi \to \pi\pi) = \eta \exp(2i\delta)$).**

We introduced three ($\rho(770)$, $\rho(1250)$ and $\rho(1550-1780)$), four (the indicated ones plus $\rho(1860-1910)$) and five (the indicated four plus $\rho(1450)$) resonances.

THE MODEL-INDEPENDENT ANALYSIS

Since in data for the P-wave $\pi\pi$ scattering a deviation from elasticity is observed in the near-threshold region of the $\omega\pi$ channel, we considered explicitly the thresholds of the $\pi\pi$ and $\omega\pi$ channels and the left-hand one at $s = 0$ in the uniformizing variable:

$$v = \frac{(m_\omega + m_{\pi^0})/2\ \sqrt{s - 4m_{\pi^+}^2} + m_{\pi^+}\ \sqrt{s - (m_\omega + m_{\pi^0})^2}}{\sqrt{s\left[((m_\omega + m_{\pi^0})/2)^2 - m_{\pi^+}^2\right]}}$$

Influence of other channels which couple to the $\pi\pi$ one is supposed to be taken into account via the background.

The resonance part of the 2-channel S-matrix element of $\pi\pi$-scattering S_{res} has no cuts on the v-plane.

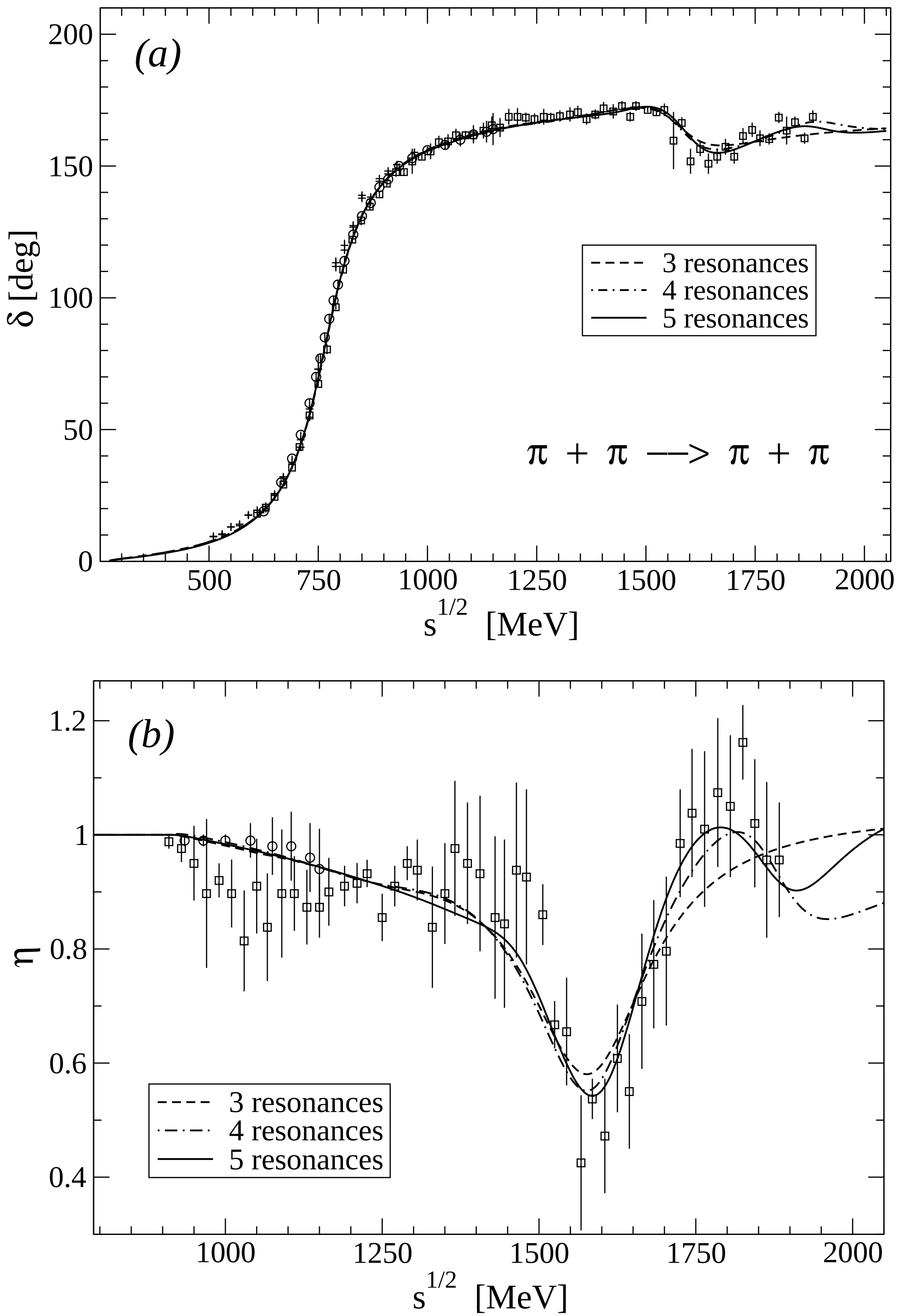
$$S_{res} = \frac{d(-v^{-1})}{d(v)}\ ,$$

where $d(v)$ represents the contribution of resonances.

Other authors have also used the parameterizations with the Jost functions in analyzing the s-wave $\pi\pi$ scattering in the one-channel approach (*J. Bohacik, H. Kühnelt, PR D **21** (1980) 1342*) and in the two-channel one (*D. Morgan, M.R. Pennington, PR D **48** (1993) 1185*).

$$S_{bg} = \exp\left[2i\left(\sqrt{\frac{s-4m_{\pi^+}^2}{s}}\right)^3\left(\alpha_0+\alpha_1\,\frac{s-s_1}{s}\,\theta(s-s_1)+\alpha_2\,\frac{s-s_2}{s}\,\theta(s-s_2)\right)\right],$$

where $\alpha_i = a_i + ib_i$, s_1 is the threshold of 4π channel noticeable in the ρ-like meson decays and s_2 is the threshold of $\rho 2\pi$ channel. Due to allowing for the left-hand branch-point at $s=0$ in the v-variable, $a_0 = b_0 = 0$. Furthermore, $b_1 = 0$ which is related to the experimental fact that the P-wave $\pi\pi$ scattering is elastic also above the 4π-channel threshold up to about the $\omega\pi^0$ threshold.



(a)
δ [deg]
200
150
100
50
0
500
750
1000
1250
1500
1750
2000
$s^{1/2}$ [MeV]
3 resonances
4 resonances
5 resonances
π + π —> π + π
(b)
η
1.2
1
0.8
0.6
0.4
1000
1250
1500
1750
2000
$s^{1/2}$ [MeV]
3 resonances
4 resonances
5 resonances

We obtained the satisfactory description with the total χ^2/NDF equal to $291.76/(183-15) = 1.74$, $278.50/(183-19) = 1.70$ and $266.14/(183-23) = 1.66$ for the case of three, four and five resonances, respectively.

The background parameters are: $a_1 = 0.0093 \pm 0.0199$, $a_2 = 0.0618 \pm 0.0305$ and $b_2 = -0.0135 \pm 0.0371$ for the three-resonance, $a_1 = 0.0017 \pm 0.2118$, $a_2 = 0.0433 \pm 0.3552$ and $b_2 = -0.0044 \pm 0.4782$ for the four-resonance, and $a_1 = 0.0256 \pm 0.0186$, $a_2 = 0.0922 \pm 0.0335$ and $b_2 = 0.0011 \pm 0.0478$ for the five-resonance descriptions. The positive sign of b_2 in the last case is more natural from the physical point of view.

Though the description can be considered, practically, as the same in all three cases, careful comparison of the obtained parameters and energy dependence of the fitted quantities suggests that the resonance $\rho(1900)$ is desired and that the $\rho(1450)$ might be also included improving slightly the description (at all events, its existence does not contradict to the data).

In Table: Pole clusters of the ρ-like states on the lower $\sqrt{s}$-half-plane (in MeV) (the conjugate poles on the upper half-plane are not shown).

	II	III	IV
$\rho(770)$	$765.8 \pm 0.6 - i(73.3 \pm 0.4)$	$778.2 \pm 9.1 - i(68.9 \pm 3.9)$	
$\rho(1250)$		$1251.4 \pm 11.3 - i(130.9 \pm 9.1)$	$1251 \pm 11.1 - i(130.5 \pm 9.2)$
$\rho(1470)$		$1469.4 \pm 10.6 - i(91 \pm 12.9)$	$1465.4 \pm 12.1 - i(99.8 \pm 15.6)$
$\rho(1600)$		$1634 \pm 20.1 - i(144.7 \pm 23.8)$	$1592.9 \pm 7.9 - i(73.7 \pm 11.7)$
$\rho(1900)$		$1882.8 \pm 24.8 - i(112.4 \pm 25.2)$	$1893 \pm 21.9 - i(93.4 \pm 19.9)$

Masses and total widths of the obtained ρ-states can be calculated from the pole positions on sheets II and IV.

	m_{res}	Γ_{tot}
$\rho(770)$	769.3±0.6	146.6±0.9
$\rho(1250)$	1257.8±11.1	261±18.3
$\rho(1470)$	1468.8±12.1	199.6±31.2
$\rho(1600)$	1594.6±8	147.4±23.4
$\rho(1900)$	1895.3±21.9	186.8±39.8

THE BREIT–WIGNER ANALYSIS

We used 5-channel Breit–Wigner forms in constructing the Jost matrix determinant $d(k_1, \cdots, k_5)$. The resonance poles and zeros in the S-matrix are generated utilizing the Le Couteur–Newton relation

$$S_{11} = \frac{d(-k_1, \cdots, k_5)}{d(k_1, \cdots, k_5)} \; ,$$

where k_1, k_2, k_3, k_4 and k_5 are the momenta of $\pi\pi$, $\pi^+\pi^-2\pi^0$, $2\pi^+2\pi^-$, $\eta 2\pi$, and $\omega\pi^0$ channels, respectively.

$$d = d_{res} d_{bg}$$

$$d_{res}(s) = \prod_r \left[M_r^2 - s - i \sum_{j=1}^{5} \rho_{rj}^3 \; R_{rj} \; f_{rj}^2 \right] ,$$

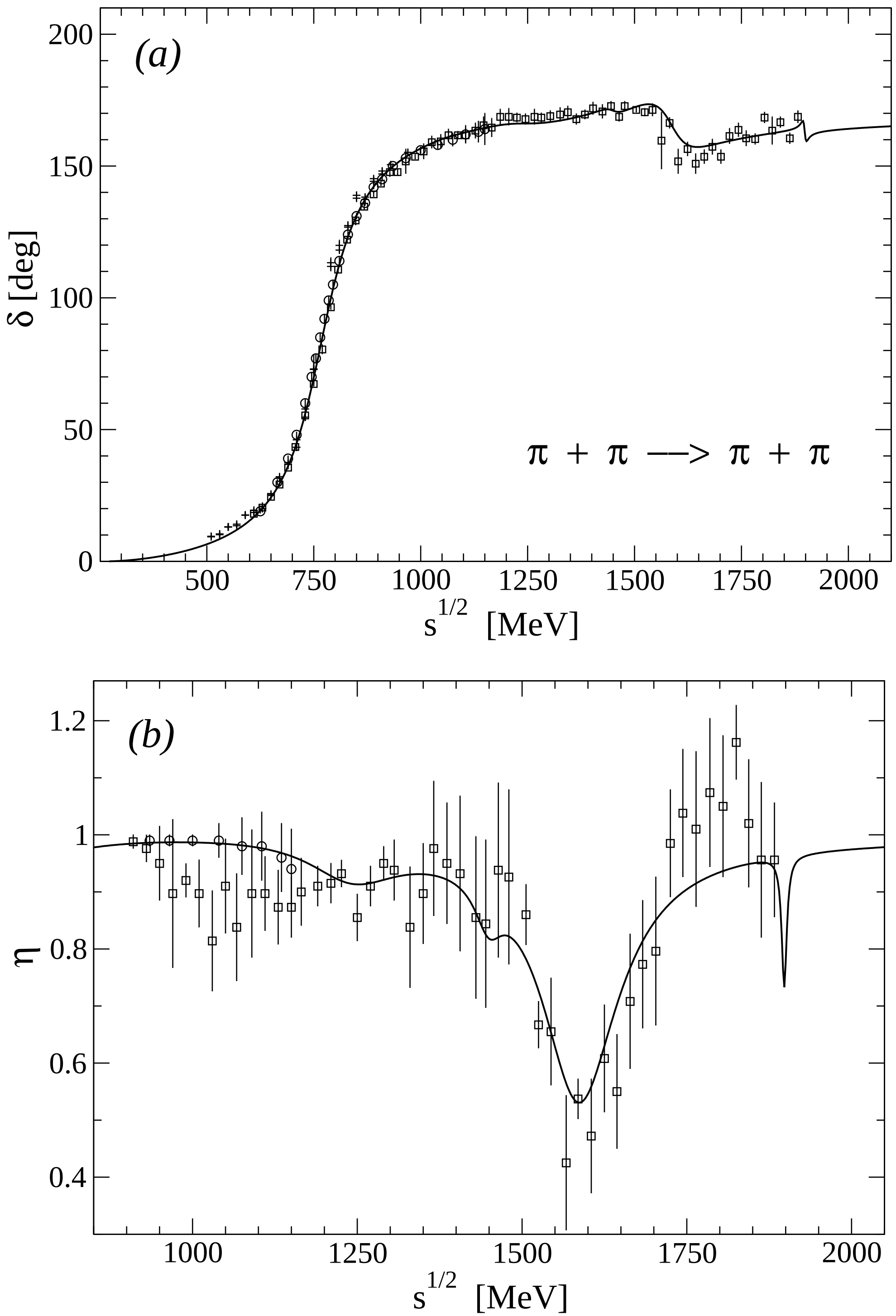
where $\rho_{rj} = k_j(s)/k_j(M_r^2)$, f_{rj}^2/M_r is the partial width of a resonance with mass M_r, and R_{rj} is a Blatt–Weisskopf barrier factor.

We took $f_{r2} = f_{r3}/\sqrt{2}$ that is well justified with a 5-10% accuracy, for example, by calculations of the ρ^0-meson decays in some variant of the chiral model (*N.N. Achasov, A.A. Kozhevnikov, PR D* ***71*** *(2005) 034015*).

The background part d_{bg} is

$$d_{bg} = \exp\left[-i\left(\sqrt{\frac{s-4m_{\pi^+}^2}{s}}\right)^3\left(\alpha_0+\alpha_1\,\frac{s-s_1}{s}\,\theta(s-s_1)\right)\right]\,,$$

where $\alpha_i = a_i + ib_i$ and s_1 is the threshold of $\rho 2\pi$ channel. We obtained equally reasonable description in all three cases: the total $\chi^2/\mathrm{NDF} = 316.21/(183-17) = 1.87$, $314.69/(183-22) = 1.92$, and $303.10/(183-27) = 1.91$ for the case of three, four, and five resonances, respectively.


(a)
δ [deg]
π + π —> π + π
s^{1/2} [MeV]
(b)
η
s^{1/2} [MeV]

The ρ-like resonance parameters (all in MeV).

State	$\rho(770)$	$\rho(1250)$	$\rho(1450)$	$\rho(1600)$	$\rho(1900)$
M	777.69±0.32	1249.8±15.6	1449.9±12.2	1587.3±4.5	1897.8±38
f_{r1}	343.8±0.73	87.7±7.4	56.9±5.4	248.2±5.2	47.3±12
f_{r2}	24.6±5.8	186.3±39.9	100.1±18.7	240.2±8.6	73.7
f_{r3}	34.8±8.2	263.5±56.5	141.6±26.5	339.7±12.5	104.3
f_{r4}		231.8±111	141.2±98	141.8±33	9
f_{r5}		231±115	150±95	108.6±40.4	10
Γ_{tot}	≈154.3	>175	>52	>168	>10

The background parameters are: $a_0 = -0.00121 \pm 0.0018$, $a_1 = -0.1005 \pm 0.011$ **and** $b_1 = 0.0012 \pm 0.006$.

In order to look at consistency of the description, we checked if the obtained formula for the $\pi\pi$-scattering amplitude gives a value of the scattering length consistent with the results of other approaches.

$a_1^1[10^{-3} m_{\pi^+}^{-3}]$	**References**	**Remarks**
33.9 ± 2.02	**This paper**	**Breit–Wigner analysis**
34	**V. Bernard et al., PL B285 (1992) 119.**	**Local NJL model**
37	**A.A. Osipov et al, arXiv:hep-ph/0603130.**	**Non-local NJL model**
37.9 ± 0.5	**I. Caprini et al., IJMP A 21 (2006) 954.**	**Roy equations using ChPT**
39.6 ± 2.4	**R. Kamiński et al., PL B551 (2003) 241.**	**Roy equations**
38.4 ± 0.8	**J.R. Peláez et al., PR D 71 (2005) 074016.**	**Forward dispersion relations**

Analysis of isoscalar-tensor sector

In analysis of the processes $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta$, we considered explicitly also the channel $(2\pi)(2\pi)$. Here it is impossible to use the uniformizing-variable method. Therefore, using the Le Couteur-Newton relations, we generate the resonance poles by some 4-channel Breit-Wigner forms.

$d(k_1, k_2, k_3, k_4)$ is taken as $\quad d = d_B d_{res}$.

$$d_{res}(s) = \prod_r \left[M_r^2 - s - i \sum_{j=1}^{4} \rho_{rj}^5 R_{rj} f_{rj}^2 \right]$$

where $\rho_{rj} = 2k_i/\sqrt{M_r^2 - 4m_j^2}$, f_{rj}^2/M_r is the partial width.

$$d_B = \exp\left[-i\sum_{n=1}^{3}\left(\frac{2k_n}{\sqrt{s}}\right)^5 (a_n + ib_n)\right].$$

$$a_1 = \alpha_{11} + \frac{s-4m_K^2}{s}\,\alpha_{12}\,\theta(s-4m_K^2) + \frac{s-s_v}{s}\,\alpha_{10}\,\theta(s-s_v)),$$

$$b_n = \beta_n + \frac{s-s_v}{s}\,\gamma_n\,\theta(s-s_v).$$

$s_v \approx 2.274$ GeV^2 is the combined threshold of channels $\eta\eta',\ \rho\rho,\ \omega\omega$.

The data for the $\pi\pi$ scattering are taken from an energy-independent analysis by B.Hyams et al. (*NP B* ***64****, 134 (1973); ibid.* ***100****,205 (1975)*).
The data for $\pi\pi \to K\overline{K}, \eta\eta$ are taken from works (*S.J.Lindenbaum, R.S.Longacre, PL B* ***274****, 492 (1992); R.S.Longacre* et al., *PL B* ***177****, 223 (1986)*).

We obtained a satisfactory description with ten resonance $f_2(1270)$, $f_2(1430)$, $f_2'(1525)$, $f_2(1580)$, $f_2(1730)$, $f_2(1810)$, $f_2(1960)$, $f_2(2000)$, $f_2(2240)$ and $f_2(2410)$ (the total $\chi^2/\mathrm{NDF} = 161.147/(168-65) \approx 1.56$) and with eleven states adding one more resonance $f_2(2020)$ which is needed in the combined analysis of processes $p\bar{p} \to \pi\pi, \eta\eta, \eta\eta'$ (*V.V.Anisovich et al., IJMP A **20, 6327 (2005)*). Description in the latter case is practically the same one as in the case of ten resonances:
the total $\chi^2/\mathrm{NDF} = 156.617/(168-69) \approx 1.58$.**

The resonance parameters for ten states (in MeV).

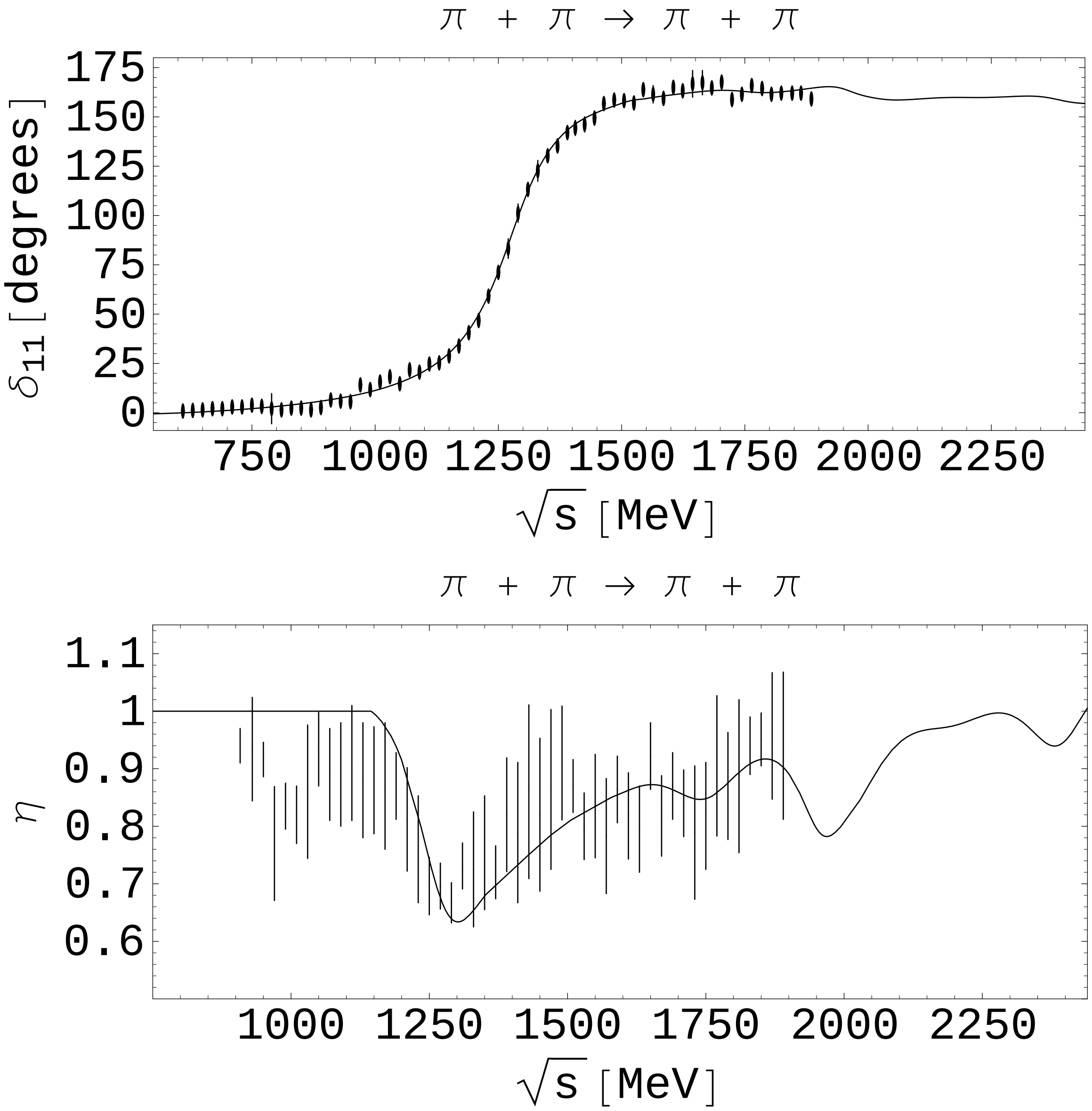
State	M	f_{r1}	f_{r2}	f_{r3}	f_{r4}	Γ_{tot}
$f_2(1270)$	1275.3±1.8	470.8±5.4	201.5±11.4	90.4±4.76	22.4±4.6	≈212
$f_2(1430)$	1450.8±18.7	128.3±45.9	562.3±142	32.7±18.4	8.2±65	>230
$f_2'(1525)$	1535±8.6	28.6±8.3	253.8±78	92.6±11.5	41.6±160	>49
$f_2(1565)$	1601.4±27.5	75.5±19.4	315±48.6	388.9±27.7	127±199	>170
$f_2(1730)$	1723.4±5.7	78.8±43	289.5±62.4	460.3±54.6	107.6±76.7	>182
$f_2(1810)$	1761.8±15.3	129.5±14.4	259±30.7	469.7±22.5	90.3±90	>177
$f_2(1960)$	1962.8±29.3	132.6±22.4	333±61.3	319±42.6	65.4±94	>119
$f_2(2000)$	2017±21.6	143.5±23.3	614±92.6	58.8±24	450.4±221	>299
$f_2(2240)$	2207±44.8	136.4±32.2	551±149	375±114	166.8±104	>222
$f_2(2410)$	2429±31.6	177±47.2	411±196.9	4.5±70.8	460.8±209	>170

For the background : $\alpha_{11} = -0.07805$, $\alpha_{12} = 0.03445$, $\alpha_{10} = -0.2295$, $\beta_1 = -0.0715$, $\gamma_1 = -0.04165$, $\beta_2 = -0.981$, $\gamma_2 = 0.736$, $\beta_3 = -0.5309$, $\gamma_3 = 0.8223$.

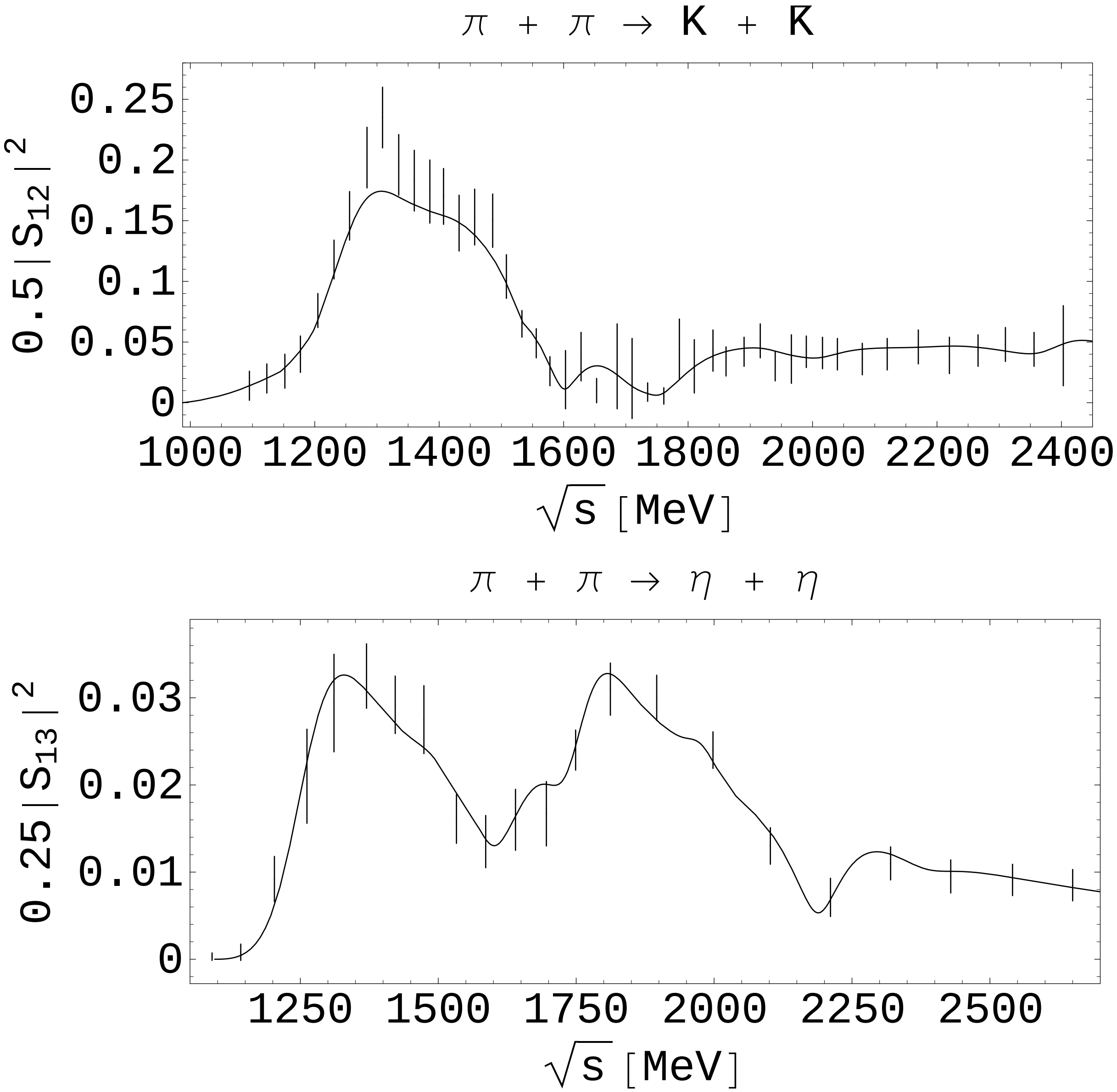
The resonance parameters for eleven states.

State	M	f_{r1}	f_{r2}	f_{r3}	f_{r4}	Γ_{tot}
$f_2(1270)$	1276.3±1.8	468.9±5.5	201.6±11.6	89.9±4.79	7.2±4.6	≈210.5
$f_2(1430)$	1450.5±18.8	128.3±45.9	562.3±144	32.7±18.6	8.2±63	>230
$f_2'(1525)$	1534.7±8.6	28.5±8.5	253.9±79	89.5±12.5	51.6±155	>49.5
$f_2(1565)$	1601.5±27.9	75.5±19.6	315±50.6	388.9±28.6	127±190	>170
$f_2(1730)$	1719.8±6.2	78.8±43	289.5±62.6	460.3±545.	108.6±76.	>182.4
$f_2(1810)$	1760±17.6	129.5±14.8	259±32.	469.7±25.2	90.3±89.5	>177.6
$f_2(1960)$	1962.2±29.8	132.6±23.3	331±61.5	319±42.8	62.4±91.3	>118.6
$f_2(2000)$	2006±22.7	155.7±24.4	169.5±95.3	60.4±26.7	574.8±211	>193
$f_2(2020)$	2027±25.6	50.4±24.8	441±196.7	58±50.8	128±190	>107
$f_2(2240)$	2202±45.4	133.4±32.6	545±150.4	381±116	168.8±103	>222
$f_2(2410)$	2387±33.3	175±48.3	395±197.7	24.5±68.5	462.8±211	>168

The background parameters are: $\alpha_{11} = -0.0755$, $\alpha_{12} = 0.0225$, $\alpha_{10} = -0.2344$, $\beta_1 = -0.0782$, $\gamma_1 = -0.05215$, $\beta_2 = -0.985$, $\gamma_2 = 0.7494$, $\beta_3 = -0.5162$, $\gamma_3 = 0.786$.



The phase shift and module of the $\pi\pi$-scattering D-wave matrix element.



The squared modules of the $\pi\pi \to K\overline{K}$ (upper figure) and $\pi\pi \to \eta\eta$ (lower figure) D-wave matrix elements.

Spectroscopic implications from the analysis

- **In the combined model-independent analysis of data on the $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta, \eta\eta'$ processes in the channel with $I^G J^{PC} = 0^+0^{++}$, an additional confirmation of the σ-meson with mass 835 MeV is obtained. This value rather accords with prediction ($m_\sigma \approx m_\rho$) on the basis of mended symmetry by S. Weinberg (*PRL* ***65****, 1177 (1990)*).**

- **Indication for $f_0(980)$ to be the $\eta\eta$ bound state is obtained. From the point of view of quark structure, this is the 4-quark state. Maybe, this is consistent somehow with arguments in favour of the 4-quark nature of $f_0(980)$ (*N.N.Achasov, NP A* ***675****, 279c (2000); M.N.Achasov* et al.*, PL B* **438***, 441 (1998); ibid.* **440***, 442 (1998)*).**

- **The $f_0(1370)$ and $f_0(1710)$ have the dominant $s\bar{s}$ component. Conclusion about the $f_0(1370)$ agrees quite well with the one drawn by the Crystal Barrel Collaboration (***C.Amsler* et al., *PL B* ***355***, *425 (1995)***) where the $f_0(1370)$ is identified as $\eta\eta$ resonance in the $\pi^0\eta\eta$ final state of the $\bar{p}p$ annihilation at rest. Conclusion about the $f_0(1710)$ is quite consistent with the experimental facts that this state is observed in $\gamma\gamma \to K_S\bar{K}_S$ (***S.Braccini, Proc. Workshop on Hadron Spectroscopy, Frascati Phys. Series* ***XV***, *53 (1999)***) and not observed in $\gamma\gamma \to \pi^+\pi^-$ (***R.Barate* et al., *PL B* ***472***, *189 (2000)***).**

- **As to the $f_0(1500)$, we suppose that it is practically the eighth component of octet mixed with a glueball being dominant in this state. Its biggest width among enclosing states tells also in behalf of its glueball nature (***V.V.Anisovich et al., NP Proc.Suppl. A*56*, 270 (1997)***).**

- **The assignment of scalar mesons to lower nonets, excluding the $f_0(980)$ as the $\eta\eta$ bound state.**
  The ground nonet: the isovector $a_0(980)$, the isodoublet $K_0^*(900)$, and $f_0(600)$ and $f_0(1370)$ as mixtures of the 8th component of octet and the SU(3) singlet. The Gell-Mann–Okubo (GM-O) formula

$$3m_{f_8}^2 = 4m_{K_0^*}^2 - m_{a_0}^2$$

  gives $m_{f_8} = 872$ MeV. ($m_\sigma = 835 \pm 14$ MeV).
  In relation for masses of nonet $m_\sigma + m_{f_0(1370)} = 2m_{K_0^*}$ the left side is about 25 % bigger than the right one.

  The next nonet: $a_0(1450)$, $K_0^*(1450)$, and $f_0(1500)$ and $f_0(1710)$. From the GM-O formula, $m_{f_8} \approx 1450$ MeV.
  In: $m_{f_0(1500)} + m_{f_0(1710)} = 2m_{K_0^*(1450)}$
  the left side is about 12 % bigger than the right one.

  Now an adequate mixing scheme should be found.

- **In the vector sector, the first ρ-like meson has the mass 1257.8±11 MeV in the model-independent analysis and 1249.8±15.6 MeV in the Breit–Wigner one. These values differ significantly from the mass (1459±11 MeV) of the first ρ-like meson cited in the PDG tables of 2006. The $\rho(1250)$ meson was discussed actively some time ago (***N.M. Budnev et al., PL B**70** (1977) 365; S.B. Gerasimov, d A.B. Govorkov, Z. Phys. C **13** (1982) 43; ibid., **29** (1985) 61*) **and later the evidence for its existence was obtained in (***D. Aston et al., NP Proc. Suppl. B**21** (1991) 105; T.S. Belozerova, V.K. Henner, Phys. Elem. Part. Atom. Nucl. **29**, part 1 (1998) 148; Yu.S. Surovtsev, P. Bydžovský, arXiv:hep-ph/0701274, to be published in Frascati Physics Series, Volume XLVI (2007); I. Yamauchi, T. Komada, Talk at the XII Int. Conf. on Hadron Spectroscopy - Hadron07, to be published in Frascati Physics Series, Volume XLVI (2007)*).

- **If the $\rho(1250)$ is interpreted as the first radial excitation of the 1^+1^{--} $q\bar{q}$ state, then it lies down well on the corresponding linear trajectory with an universal slope on the (n, M^2) plane (n is the radial quantum number of the $q\bar{q}$ state) (*A.V. Anisovich et al., PR D **62** (2000) 051502*), whereas the $\rho(1450)$ turns out to be considerably higher than this trajectory. The $\rho(1250)$ and the isodoublet $K^*(1410)$ are well located to the octet of first radial excitations. The mass of the latter should be by about 150 MeV larger than the mass of the former. Then the GM-O formula**

$$3m^2_{\omega'_8} = 4m^2_{K^{*\prime}} - m^2_{\rho'}$$

gives the value $m_{\omega'_8} = 1460$ MeV, that is fairly good compatible with the mass of the first ω-like meson $\omega(1420)$, for which one obtains the values in range 1350-1460 MeV (PDG 2006).

- **Existence of the $\rho(1450)$ (along with $\rho(1250)$) does not contradict to the data. In the $q\bar{q}$ picture, it might be the first $^3D_1$ state with, possibly, the isodoublet $K^*(1680)$ in the corresponding octet. From the GM-O formula, we should obtain the value 1750 MeV for the mass of the eighth component of this octet. This corresponds to one of the observations of the second ω-like meson with masses from 1606 to 1840 MeV that is cited in the PDG tables under the $\omega(1650)$.**

- **The third ρ-like meson has the mass about 1600 MeV rather than 1720 MeV cited in the PDG tables.**

- **As to the $\rho(1900)$, in this energy region there are practically no data on the P-wave of $\pi\pi$ scattering. The model-independent analysis testifies in favour of existence of this state, whereas the Breit–Wigner analysis gives the same description with and without the $\rho(1900)$.**

- **The suggested picture for the first two ρ-like mesons is consistent with predictions of the quark model (*E. van Beveren et al., PR D* ***27*** *(1983) 1527*). In (*S.B. Gerasimov, d A.B. Govorkov, Z. Phys. C* ***13*** *(1982) 43; ibid.,* ***29*** *(1985) 61*) the discussed mass spectrum for radially excited ρ- and K^*-mesons was obtained using rather simple mass operator. If the existence of the $\rho(1250)$ is confirmed, some quark potential models, *e.g.*, in (*S. Godfrey, N. Isgur, PR D* ***32*** *(1985) 189*), will require substantial revisions, because the first ρ-like meson is usually predicted about 200 MeV higher than this state. To the point, the first K^*-like meson is obtained in the indicated quark model at 1580 MeV, whereas the corresponding very well established resonance has the mass of only 1410 MeV.**

- **As to the tensor sector, we carried out two analysis – without and with the $f_2(2020)$. We do not obtain $f_2(1640)$, $f_2(1910)$ and $f_2(2150)$, however, we see $f_2(1450)$ and $f_2(1730)$ which are related to the statistically-valued experimental points.**

- **Usually one assigns to the ground tensor nonet the states $f_2(1270)$ and $f_2'(1525)$. To the 2nd nonet, one could assign $f_2(1600)$ and $f_2(1760)$ though for now the isodoublet member is not discovered. If $a_2(1730)$ is the isovector of this octet and if $f_2(1600)$ is almost its 8th component, then, from the GM-O formula, we expect this isodoublet mass at about 1633 MeV. Then the relation for masses of nonet would be fulfilled with a 3% accuracy. V.M.Karnaukhov et al. (*Yad.Fiz.* ***63**, 652 (2000)*) observed the strange isodoublet with yet indefinite remaining quantum numbers and with mass 1629 ± 7 MeV in the mode $K_s^0\pi^+\pi^-$. This state might be the tensor isodoublet of the 2nd nonet.**

- **The states $f_2(1963)$ and $f_2(2207)$ together with the isodoublet $K_2^*(1980)$ could be put into the third nonet. Then in the relation for masses of nonet**

$$M_{f_2(1963)} + M_{f_2(2207)} = 2M_{K_2^*(1980)},$$

the left-hand side is only 5.3 % bigger than the right-hand one.

If one consider $f_2(1963)$ as the eighth component of octet, the GM-O formula

$$M_{a_2}^2 = 4M_{K_2^*(1980)}^2 - 3M_{f_2(1963)}^2$$

gives $M_{a_2} = 2030$ MeV. This value coincides with the one for a_2-meson obtained in works: *A.V.Anisovich et al., PL B **452**, 173 (1999); ibid., **452**, 187 (1999); ibid., **517**, 261 (2001).*. **This state is interpreted as a second radial excitation of the 1^-2^{++}-state on the basis of consideration of the a_2 trajectory on the (n, M^2) plane (***V.V.Anisovich et al.. IJMP A **20**, 6327 (2005)***).**

- **As to $f_2(2000)$, the presence of the $f_2(2020)$ in the analysis with eleven resonances helps to interpret $f_2(2000)$ as the glueball. In the case of ten resonances, the ratio of the $\pi\pi$ and $\eta\eta$ widths is in the limits obtained in Ref.(*V.V.Anisovich et al., IJMP A **20**, 6327 (2005)*) for the tensor glueball on the basis of the 1/N-expantion rules. However, the $K\overline{K}$ width is too large for the glueball. At practically the same description of processes with the consideration of eleven resonances as in the case of ten, their parameters have varied a little, except for the ones for $f_2(2000)$ and $f_2(2410)$. Mass of the latter has decreased by about 40 MeV. As to $f_2(2000)$, its $K\overline{K}$ width has changed significantly. Now all the obtained ratios of the partial widths are in the limits corresponding to the glueball.**

- **The question of interpretation of the $f_2(1450)$, $f_2(1730)$, $f_2(2020)$ and $f_2(2410)$ is open.**

APPENDIXES

The S-matrix is determined on the 4- and 8-sheeted Riemann surfaces for the 2- and 3-channel cases, respectively. The matrix elements $S_{\alpha\beta}$, where $\alpha, \beta = 1, 2, 3$ denote channels, have the right-hand cuts along the real axis of the s complex plane (s is the invariant total energy squared), starting with s_i ($i = 1, 2, 3$), and the left-hand cuts.

The Riemann-surface sheets are numbered according to the signs of analytic continuations of the channel momenta

$$k_i = \sqrt{s - s_i}/2 \qquad (i = 1, 2, 3) \qquad\qquad \text{as follows:}$$

	I	II	III	IV	V	VI	VII	VIII
Imk_1	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
Imk_2	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
Imk_3	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$

The resonance representations on the Riemann surfaces are obtained with the help of formulas from (*KMS, 96)*), expressing analytic continuations of the matrix elements to unphysical sheets in terms of those on sheet I that have only the resonances zeros (beyond the real axis), at least, around the physical region.

In the 2-channel case, we obtain 3 types of resonances described by a pair of conjugate zeros on sheet I: (a) in S_{11}, (b) in S_{22}, (c) in each of S_{11} and S_{22}.

In the 3-channel case, we obtain 7 types of resonances corresponding to 7 possible situations when there are resonance zeros on sheet I only in S_{11} – (a); S_{22} – (b); S_{33} – (c); S_{11} and S_{22} – (d); S_{22} and S_{33} – (e); S_{11} and S_{33} – (f); S_{11}, S_{22}, and S_{33} – (g).
A resonance of every type is represented by a pair of complex-conjugate clusters (of poles and zeros on the Riemann surface). Note that whereas cases (a), (b) and (c) can be simply related to the representation of resonances by Breit-Wigner forms, cases (d), (e), (f) and (g) practically are lost at that description. The cluster kind is related to the nature of state. For example, if we consider the $\pi\pi$, $K\overline{K}$ and $\eta\eta$ channels, then a resonance, coupled relatively more strongly to the $\pi\pi$ channel than to the $K\overline{K}$ and $\eta\eta$ ones is described by the cluster of type (a). If the resonance is coupled more strongly to the $K\overline{K}$ and $\eta\eta$ channels than to the $\pi\pi$, it is represented by the cluster of type (e) (say, the state with the dominant $s\bar{s}$ component).

The flavour singlet (*e.g.*, glueball) must be represented by the cluster of type (g) (of type (c) in the 2-channel consideration) as a necessary condition for the ideal case, if this state lies above the thresholds of considered channels.

We can distinguish, in a model-independent way, a bound state of colourless particles (*e.g.*, $K\overline{K}$ molecule) and a $q\bar{q}$ bound state. Just as in the 1-channel case, the existence of the particle bound-state means the presence of a pole on the real axis under the threshold on the physical sheet, so in the 2-channel case, the existence of the particle bound-state in channel 2 ($K\overline{K}$ molecule) that, however, can decay into channel 1 ($\pi\pi$ decay), would imply the presence of a pair of complex conjugate poles on sheet II under the second-channel threshold without the corresponding shifted pair of poles on sheet III.

In the 3-channel case, the bound-state in channel 3 ($\eta\eta$) that, however, can decay into channels 1 ($\pi\pi$ decay) and 2 ($K\overline{K}$ decay), is represented by the pair of complex conjugate poles on sheet II and by a shifted poles on sheet III under the $\eta\eta$ threshold without the corresponding poles on sheets VI and VII. This test (*D. Morgan, M.R. Pennington, PR D 48, 1185 (1993); KMS, 96*) is the multichannel analogue of the known Castillejo–Dalitz–Dyson poles in the one-channel case. According to this test, earlier in (*KMS, 96)*), the interpretation of the $f_0(980)$ state as the $K\overline{K}$ molecule has been rejected because this state is represented by the cluster of type (a) in the 2-channel analysis of processes $\pi\pi \to \pi\pi, K\overline{K}$.

We use the Le Couteur-Newton relations (*K.J.LeCouteur, Proc.Roy.Soc. A **256, 115 (1960); R.G.Newton, J.Math.Phys. **2**, 188 (1961); M.Kato, Ann.Phys. **31**, 130 (1965)*). They express the S-matrix elements of all coupled processes in terms of the Jost matrix determinant $d(k_1, \cdots, k_n)$ that is a real analytic function with the only square-root branch-points at $k_i = 0$.**

The important branch points, corresponding to the thresholds of the coupled channels and to the crossing ones, are taken into account in a proper uniformizing variable.

On the uniformization plane, the pole-cluster representation of a resonance is a good one.

the $f_0(600)$ is described by the cluster of type (a);
$f_0(1370)$, type (c); $f_0(1500)$, type (g); $f_0(1710)$, type (b);
the $f_0(980)$ is represented only by the pole on sheet II and shifted pole on sheet III.

The combined description of processes $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta'$ (variant II) is even better due to the more detailed representation of the background:
the $f_0(600)$ is described by the cluster of type (a′);
$f_0(1370)$, type (b′); $f_0(1500)$, type (d′); $f_0(1710)$, type (c′).

For subsequent conclusions, let us mention the results for coupling constants from our previous 2-channel analysis (*Yu.S.Surovtsev, D.Krupa, M.Nagy, EPJ A* ***15**, 409 (2002)*): g_1 is the coupling constant with $\pi\pi$; g_2, with $K\overline{K}$.**

	$f_0(600)$	$f_0(980)$	$f_0(1370)$	$f_0(1500)$
g_1, GeV	0.652 ± 0.065	0.167 ± 0.05	0.116 ± 0.03	0.657 ± 0.113
g_2, GeV	0.724 ± 0.1	0.445 ± 0.031	0.99 ± 0.05	0.666 ± 0.15

The $f_0(980)$ and the $f_0(1370)$ are coupled essentially more strongly to the $K\overline{K}$ system than to the $\pi\pi$ one, *i.e.*, they have a dominant $s\bar{s}$ component. The $f_0(1500)$ has the approximately equal coupling constants with the $\pi\pi$ and $K\overline{K}$, which apparently could point to its dominant glueball component. In the 2-channel case, $f_0(1710)$ is represented by the cluster corresponding to a state with the dominant $s\bar{s}$ component.

Comparison of the data for the phase shift below the $K\overline{K}$ threshold (*P. Estabrooks and A.D. Martin, NP B **79, 301 (1974)*) with those of (*S.D. Protopopescu et al., PR D **7**, 1279 (1973); B. Hyams et al., NP B **64**, 134 (1973)*) shows that the former data are systematically by 1°-5° larger than the latter, except for two points of (*S.D. Protopopescu et al.*) at 710 and 730 MeV, which lie by about 2° higher than the corresponding points of the former work. These two data points were omitted in the analyses. Since we do not know an energy dependence of the remarked deviations of the data points, we assume a constant systematic error that must be determined in the combined analysis of data. This simple assumption about the constant systematic error is the least offensive intervention to the data because it does not change their character, nevertheless makes the used data compatible with each other.**

We have obtained the satisfactory description with χ^2/NDF and with the indicated systematic error equal respectively to $291.76/(183-15) = 1.74$ and $-1.97^\circ \pm 0.19^\circ$ for the case of three resonances, $278.50/(183-19) = 1.70$ and $-1.90^\circ \pm 0.2^\circ$ for four resonances, and $266.14/(183-23) = 1.66$ and $-1.87^\circ \pm 0.19^\circ$ for five resonances.

Blatt–Weisskopf barrier factor conditioned by the resonance spins. For the vector particle this factor has the form:

$$R_{rj} = \frac{1 + \frac{1}{4}(\sqrt{M_r^2 - 4m_j^2}\ r_{rj})^2}{1 + \frac{1}{4}(\sqrt{s - 4m_j^2}\ r_{rj})^2}$$

where r_{rj} is a radius of the j-channel decay. In our analysis, $r_{rj} = 0.7035$ fm identical for all resonances in all channels.

a_1 and b_1 take into account also influence of other

channels opened at higher energies than the $\rho 2\pi$ threshold. The parameter b_0 is set to zero in this analysis, similarly as in the model-independent approach.

The analysis is performed taking into account three, four, and five resonances.

The Blatt–Weisskopf barrier factor for a tensor particle is

$$R_{rj} = \frac{9 + \frac{3}{4}(\sqrt{M_r^2 - 4m_j^2}\ r_{rj})^2 + \frac{1}{16}(\sqrt{M_r^2 - 4m_j^2}\ r_{rj})^4}{9 + \frac{3}{4}(\sqrt{s - 4m_j^2}\ r_{rj})^2 + \frac{1}{16}(\sqrt{s - 4m_j^2}\ r_{rj})^4}$$

with radii of 0.943 fm for all resonances in all channels, except for $f_2(1270)$ and $f_2(1960)$ for which they are: for $f_2(1270)$, 1.498, 0.708 and 0.606 fm in channels $\pi\pi$, $K\overline{K}$ and $\eta\eta$, for $f_2(1960)$, 0.296 fm in channel $K\overline{K}$.

under natural assumption that the parameters of spin-spin splitting in radial excitations as compared to the splitting in the ground states change by a factor proportional to the ratio of the corresponding wave functions “at zero”.

Finally we have $f_2(1450)$ and $f_2(1730)$ which are neither $q\bar{q}$ states nor glueballs. Since one predicts that masses of the lightest $q\bar{q}g$ hybrids are bigger than the ones of lightest glueballs, maybe, these states are the 4-quark ones. Of course, assumption of this possibility presupposes an existence of the scalar 4-quark states at lower energies which are not seen in analysis. One can think that these states are a part of the background due to their very large widths.