Hadronic light-by-light scattering in the muon $g-2$: impact of proposed measurements of the $\pi^0 \to \gamma\gamma$ decay width and the $\gamma^*\gamma \to \pi^0$ transition form factor with the KLOE-2 experiment

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Outline

- Hadronic light-by-light scattering in the muon $g-2$
- Pion-exchange versus pion-pole: off-shell versus on-shell form factors
- Impact of KLOE-2 measurements
- Conclusions

MESON 2012, May 31 - June 5, 2012, Cracow, Poland

Our work

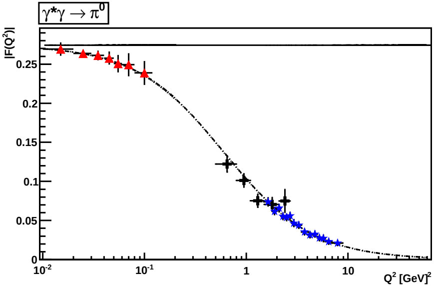
On the possibility to measure the $\pi^0 \to \gamma\gamma$ decay width and the $\gamma^*\gamma \to \pi^0$ transition form factor with the KLOE-2 experiment

D. Babusci, H. Czyż, F. Gonnella, S. Ivashyn, M. Mascolo, R. Messi,
D. Moricciani, A. Nyffeler, G. Venanzoni and the KLOE-2 Collaboration

Eur. Phys. J. **C72**, 1917 (2012) [arXiv:1109.2461 [hep-ph]]

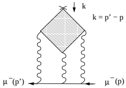
Within 1 year of data taking, collecting 5 fb^{-1}, KLOE-2 will be able to measure:

- $\Gamma_{\pi \to \gamma\gamma}$ to 1% statistical precision.
- $\gamma^*\gamma \to \pi^0$ transition form factor $F(Q^2)$ in the region of very low, space-like momenta
  $0.01\ \mathrm{GeV}^2 \leq Q^2 \leq 0.1\ \mathrm{GeV}^2$
  with a statistical precision of less than 6% in each bin.
  KLOE-2 can (almost) directly measure slope of form factor at origin.



Simulation of KLOE-2 measurement of $F(Q^2)$ (red triangles).
Solid line: $F(0)$ given by chiral anomaly.
Dashed line: form factor according to on-shell LMD+V model (Knecht, Nyffeler, EPJC '01).
CELLO (black crosses) and CLEO (blue stars) data at higher Q^2.

Hadronic light-by-light scattering: Summary of selected results



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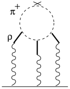
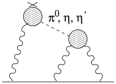
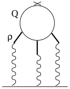


$+ \cdots +$



$+ \cdots +$ Exchange of other resonances ($f_0, a_1, \ldots$) $+$



$+ \cdots$

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Relevant scales in $\langle VVVV \rangle$ (off-shell !): $0 - 2$ GeV, i.e. much larger than m_μ !

No direct relation to experimental data $\rightarrow$ need hadronic model (or lattice QCD)

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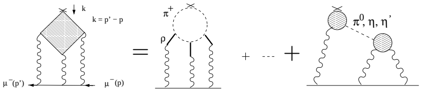

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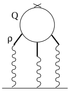

+ ···

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Contribution to $a_\mu \times 10^{11}$:

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN:	+116 (40)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
GFW:	+217 (91)		+81 (12)		+136 (59)
GdR:	+150 (3)		+68 (3)		+82 (6)
		ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09; N,JN = Nyffeler '09; Jegerlehner, Nyffeler '09; GFW = Goecke, Fischer, Williams '11; GdR = Greynat, de Rafael '12 (error only reflects variation $M_Q = 240 \pm 10$ MeV, 20%-30% systematic error)

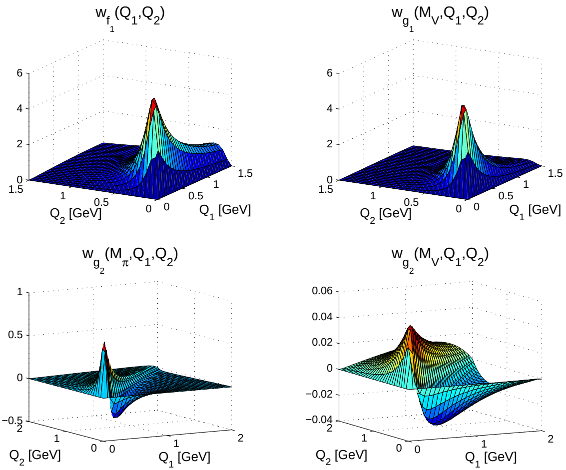
Recall (in units of 10^{-11}): δa_μ(had. VP) ≈ 45; δa_μ(exp [BNL]) $= 63$; δa_μ(future exp) $= 15$

Relevant momentum regions in $a_\mu^{\mathrm{LbyL};\pi^0}$

In Knecht, Nyffeler '02, a 2-dimensional integral representation was derived for a certain class (VMD-like) of form factors (schematically):

$$a_\mu^{\mathrm{LbyL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2)\, f_i(Q_1, Q_2)$$

with universal weight functions w_i. Dependence on form factors resides in the f_i.

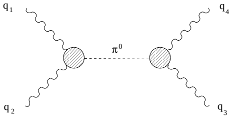


Relevant momentum regions around 0.25 – 1.25 GeV. As long as form factors in different models lead to damping, we expect comparable results for $a_\mu^{\mathrm{LbyL};\pi^0}$, at the level of 20%.

Jegerlehner, Nyffeler '09 derived 3-dimensional integral representation for general form factors. Integration over $Q_1^2, Q_2^2, \cos\theta$, where $Q_1 \cdot Q_2 = |Q_1||Q_2|\cos\theta$.

Pion-pole in $\langle VVVV \rangle$ versus pion-exchange in had. LbyL in a_μ

To uniquely identify contribution of exchanged neutral pion π^0 in Green's function $\langle VVVV \rangle$, we need to pick out pion-pole:



+ crossed diagrams

$$\lim_{(q_1+q_2)^2 \to m_\pi^2} ((q_1+q_2)^2 - m_\pi^2) \langle VVVV \rangle$$

Residue of pole: on-shell vertex function $\langle 0|VV|\pi\rangle \to$ on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

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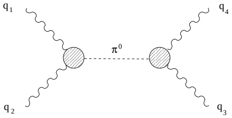

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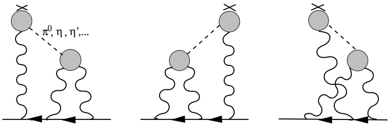
But in contribution to the muon $g-2$, we have to evaluate Feynman diagrams, integrating over the photon momenta with exchanged off-shell pions.
For all pseudoscalars:



Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}$ where $\mathrm{PS} = \pi^0, \eta, \eta', \pi^{0\prime}, \ldots$
Off-shell form factors are either inserted "by hand" starting from constant, pointlike Wess-Zumino-Witten (WZW) form factor or using e.g. some resonance Lagrangian.

Similar statements apply for exchanges (or loops) of other resonances.

Off-shell pion form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ from $\langle VVP \rangle$

- Following Bijnens, Pallante, Prades '95, '96; Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, we can define off-shell form factor for π^0:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x+q_2\cdot y)}\, \langle 0 | T\{j_\mu(x) j_\nu(y) P^3(0)\} | 0 \rangle$$

$$= \quad \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \frac{i\langle \overline{\psi}\psi \rangle}{F_\pi}\, \frac{i}{(q_1+q_2)^2 - m_\pi^2}\, \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) + \ldots$$

Up to small mixing effects of P^3 with η and η' and neglecting exchanges of heavier states like $\pi^{0\prime}, \pi^{0\prime\prime}, \ldots$

j_μ = light quark part of the electromagnetic current: $j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x)$

$$\psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad \hat{Q} = \mathrm{diag}(2,-1,-1)/3$$

$P^3 = \overline{\psi} i\gamma_5 \frac{\lambda^3}{2}\psi = \left(\overline{u} i\gamma_5 u - \overline{d} i\gamma_5 d\right)/2, \quad \langle \overline{\psi}\psi \rangle$ = single flavor quark condensate

Bose symmetry: $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_2^2, q_1^2)$

- Note: for off-shell pions, instead of $P^3(x)$, we could use any other suitable interpolating field, like $(\partial^\mu A_\mu^3)(x)$ or even an elementary pion field $\pi^3(x)$!

On-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ and transition form factor $F(Q^2)$

- On-shell $\pi^0\gamma^*\gamma^*$ form factor between an on-shell pion and two off-shell photons:

$$i\int d^4x\, e^{iq_1\cdot x}\langle 0|T\{j_\mu(x)j_\nu(0)\}|\pi^0(q_1+q_2)\rangle = \varepsilon_{\mu\nu\rho\sigma}\,q_1^\rho q_2^\sigma\,\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)$$

  Relation to off-shell form factor:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) \equiv \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2,q_1^2,q_2^2)$$

  Form factor for real photons is related to $\pi^0 \to \gamma\gamma$ decay width:

$$\mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(q_1^2=0,q_2^2=0) = \frac{4}{\pi\alpha^2 m_\pi^3}\Gamma_{\pi^0\to\gamma\gamma}$$

  Often normalization with chiral anomaly is used:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi}$$

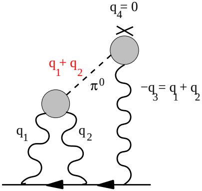
- Pion-photon transition form factor:

$$F(Q^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, q_2^2=0), \qquad Q^2 \equiv -q_1^2$$

  Note that $q_2^2 = 0$, but $\vec{q}_2 \neq \vec{0}$ for on-shell photon !

Pion-exchange versus pion-pole contribution to $a_\mu^{\mathrm{LbyL};\pi^0}$

- Off-shell form factors have been used to evaluate the pion-exchange contribution in Bijnens et al '96, Hayakawa et al '96, '98. "Rediscovered" by Jegerlehner in '07, '08. Consider diagram:



$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \;\times\; \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

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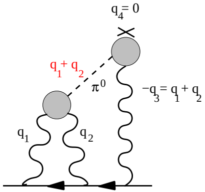

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

- On the other hand, Knecht, Nyffeler '02 used on-shell form factors:

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$$

- But form factor at external vertex $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes ! Often the following misleading notation was used

  $\mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$

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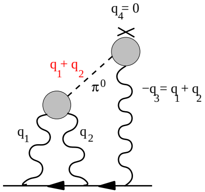

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  At external vertex identification with transition form factor was made (wrongly !).

- Melnikov, Vainshtein '04 had already observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \ \times \ \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, m_\pi^2, 0)$$

  i.e. a constant form factor at the external vertex given by the WZW term.

Pion-exchange versus pion-pole contribution to $a_\mu^{\mathrm{LbyL};\pi^0}$ (continued)

- However, this prescription will only yield the so-called pion-pole contribution and not the full pion-exchange contribution !
- In general, any evaluation e.g. using some resonance Lagrangian, will lead to off-shell form factors at both the vertices in the Feynman integral.
- Strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell. Only in some specific model where pions appear as propagating fields can one identify the contribution from off-shell pions.

In the numerical results later, we will denote by

- (JN): pion-exchange contribution with off-shell pion form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ at both vertices.
- (MV): pion-pole contribution with on-shell pion form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ at one vertex and constant form factor (WZW) at external vertex.

KLOE-2 impact on $a_\mu^{\mathrm{LbyL};\pi^0}$

- Value of $a_\mu^{\mathrm{LbyL};\pi^0}$ is currently obtained using various hadronic models.
- Any experimental information on the relevant form factors can therefore help to check the consistency of models and reduce the error.
- As stressed before, what enters in $a_\mu^{\mathrm{LbyL};\pi^0}$ is the fully off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ (vertex function).
- A measurement of the transition form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, q^2, 0)$ can, in general, only be sensitive to a subset of the model parameters and, in general, does not allow to reconstruct the full off-shell form factor.
- Good description for transition form factor is only necessary, not sufficient, in order to uniquely determine $a_\mu^{\mathrm{LbyL};\pi^0}$.
- From one model to another, uncertainty of $a_\mu^{\mathrm{LbyL};\pi^0}$ related to the off-shell pion can be very different. Complete error on $a_\mu^{\mathrm{LbyL};\pi^0}$ should take into account model dependence.

KLOE-2 impact on $a_\mu^{\mathrm{LbyL};\pi^0}$ (continued)

For illustration, but not to present some new "realistic" estimate, we will study the impact of the KLOE-2 measurements on two models:

- VMD (off-shell): has only two parameters.

  Other models with very few parameters are constituent quark models or holographic models (AdS/QCD).

- LMD+V (off-shell) (Knecht, Nyffeler, EPJC '01): has many poorly constrained parameters.

  Including the uncertainties related to the off-shellness of the pion, which dominate the final error, one obtains the estimate:

  $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$

  (Nyffeler '09; Jegerlehner, Nyffeler '09).

The VMD form factor

Vector Meson Dominance:

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = \frac{N_C}{12\pi^2 F_\pi} \frac{M_V^2}{q_1^2 - M_V^2} \frac{M_V^2}{q_2^2 - M_V^2}$$

on-shell = off-shell form factor !

Only two model parameters even for off-shell form factor: F_π and M_V

Transition form factor:

$$F^{\mathrm{VMD}}(Q^2) = \frac{N_C}{12\pi^2 F_\pi} \frac{M_V^2}{Q^2 + M_V^2}$$

The LMD+V form factor (off-shell)

Knecht, Nyffeler, EPJC '01; Nyffeler '09

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ in large-N_C QCD in chiral limit with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all leading (and some subleading) QCD short-distance constraint from Operator Product Expansion (OPE)
- Reproduces Brodsky-Lepage (BL): $\lim_{Q^2 \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$

  (OPE and BL cannot be fulfilled simultaneously with only one vector resonance)
- Normalized to decay width $\Gamma_{\pi^0 \to \gamma\gamma}$

Off-shell LMD+V form factor:

$$
\begin{aligned}
\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}(q_3^2, q_1^2, q_2^2) &= -\frac{F_\pi}{3} \frac{q_1^2\, q_2^2\,(q_1^2 + q_2^2 + q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)(q_1^2 - M_{V_2}^2)(q_2^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2)} \\
P_H^V(q_1^2, q_2^2, q_3^2) &= h_1\,(q_1^2 + q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\,(q_1^2 + q_2^2)\, q_3^2 + h_4\, q_3^4 \\
&\quad + h_5\,(q_1^2 + q_2^2) + h_6\, q_3^2 + h_7 \\
q_3^2 &= (q_1 + q_2)^2
\end{aligned}
$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV

Free parameters: h_i

The LMD+V form factor (on-shell)

On-shell LMD+V form factor:

$$\begin{aligned}
&\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\mathrm{LMD+V}}(q_1^2, q_2^2) \\
&= \quad -\frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2) + h_1\, (q_1^2 + q_2^2)^2 + \bar{h}_2\, q_1^2\, q_2^2 + \bar{h}_5\, (q_1^2 + q_2^2) + \bar{h}_7}{(q_1^2 - M_{V_1}^2)\, (q_1^2 - M_{V_2}^2)\, (q_2^2 - M_{V_1}^2)\, (q_2^2 - M_{V_2}^2)}
\end{aligned}$$

$$\begin{aligned}
\bar{h}_2 &= h_2 + m_\pi^2 \\
\bar{h}_5 &= h_5 + h_3 m_\pi^2 \\
\bar{h}_7 &= h_7 + h_6 m_\pi^2 + h_4 m_\pi^4
\end{aligned}$$

Transition form factor:

$$F^{\mathrm{LMD+V}}(Q^2) = -\frac{F_\pi}{3} \frac{1}{M_{V_1}^2 M_{V_2}^2} \frac{h_1 Q^4 - \bar{h}_5 Q^2 + \bar{h}_7}{(Q^2 + M_{V_1}^2)(Q^2 + M_{V_2}^2)}$$

- $h_1 = 0$ in order to reproduce Brodsky-Lepage behavior.
- Can treat h_1 as free parameter to fit the BABAR data, but the form factor does then not vanish for $Q^2 \to \infty$, if $h_1 \neq 0$.
  As pointed out by Dorokhov '10, this violates the Terazawa-West inequality $|F(Q^2)| \leq 1/Q$ which follows from unitarity ('72, '73).

Form factor $F(Q^2)$: data sets and normalization

Data sets used for fits:

$$\begin{aligned}
\text{A0}: &\quad \text{CELLO, CLEO, PDG} \\
\text{A1}: &\quad \text{CELLO, CLEO, PrimEx} \\
\text{A2}: &\quad \text{CELLO, CLEO, PrimEx, KLOE-2} \\
\\
\text{B0}: &\quad \text{CELLO, CLEO, BABAR, PDG} \\
\text{B1}: &\quad \text{CELLO, CLEO, BABAR, PrimEx} \\
\text{B2}: &\quad \text{CELLO, CLEO, BABAR, PrimEx, KLOE-2}
\end{aligned}$$

Normalization for $F(0)$:

- $\Gamma^{\mathrm{PDG}}_{\pi^0 \to \gamma\gamma} = 7.74 \pm 0.48$ eV (6.2% precision) for current PDG value
- $\Gamma^{\mathrm{PrimEx}}_{\pi^0 \to \gamma\gamma} = 7.82 \pm 0.22$ eV (2.8% precision) from PrimEx experiment
- $\Gamma^{\mathrm{KLOE-2}}_{\pi^0 \to \gamma\gamma} = 7.73 \pm 0.08$ eV (1% precision) for the KLOE-2 simulation

As noted in Nyffeler, PoS '09, the uncertainty in the normalization of the form factor was not taken into account in most evaluations of $a_\mu^{\mathrm{LbyL};\pi^0}$ (with the exception later of Dorokhov et al. '11).

In most papers, simply $F_\pi = 92.4$ MeV is used without any error attached to it. Value is close to $F_\pi = (92.20 \pm 0.14)$ MeV obtained from $\pi^+ \to \mu^+ \nu_\mu(\gamma)$.

Fitting the models

Model	Data	$\chi^2/d.o.f.$		Parameters	
VMD	A0	6.6/19	$M_V = 0.778(18)$ GeV	$F_\pi = 0.0924(28)$ GeV	
VMD	A1	6.6/19	$M_V = 0.776(13)$ GeV	$F_\pi = 0.0919(13)$ GeV	
VMD	A2	7.5/27	$M_V = 0.778(11)$ GeV	$F_\pi = 0.0923(4)$ GeV	
VMD	B0	77/36	$M_V = 0.829(16)$ GeV	$F_\pi = 0.0958(29)$ GeV	
VMD	B1	78/36	$M_V = 0.813(8)$ GeV	$F_\pi = 0.0925(13)$ GeV	
VMD	B2	79/44	$M_V = 0.813(5)$ GeV	$F_\pi = 0.0925(4)$ GeV	
LMD+V, $h_1 = 0$	A0	6.5/19	$\bar{h}_5 = 6.99(32)$ GeV^4	$\bar{h}_7 = -14.81(45)$ GeV^6	
LMD+V, $h_1 = 0$	A1	6.6/19	$\bar{h}_5 = 6.96(29)$ GeV^4	$\bar{h}_7 = -14.90(21)$ GeV^6	
LMD+V, $h_1 = 0$	A2	7.5/27	$\bar{h}_5 = 6.99(28)$ GeV^4	$\bar{h}_7 = -14.83(7)$ GeV^6	
LMD+V, $h_1 = 0$	B0	65/36	$\bar{h}_5 = 7.94(13)$ GeV^4	$\bar{h}_7 = -13.95(42)$ GeV^6	
LMD+V, $h_1 = 0$	B1	69/36	$\bar{h}_5 = 7.81(11)$ GeV^4	$\bar{h}_7 = -14.70(20)$ GeV^6	
LMD+V, $h_1 = 0$	B2	70/44	$\bar{h}_5 = 7.79(10)$ GeV^4	$\bar{h}_7 = -14.81(7)$ GeV^6	
LMD+V, $h_1 \neq 0$	A0	6.5/18	$\bar{h}_5 = 6.90(71)$ GeV^4	$\bar{h}_7 = -14.83(46)$ GeV^6	$h_1 = -0.03(18)$ GeV^2
LMD+V, $h_1 \neq 0$	A1	6.5/18	$\bar{h}_5 = 6.85(67)$ GeV^4	$\bar{h}_7 = -14.91(21)$ GeV^6	$h_1 = -0.03(17)$ GeV^2
LMD+V, $h_1 \neq 0$	A2	7.5/26	$\bar{h}_5 = 6.90(64)$ GeV^4	$\bar{h}_7 = -14.84(7)$ GeV^6	$h_1 = -0.02(17)$ GeV^2
LMD+V, $h_1 \neq 0$	B0	18/35	$\bar{h}_5 = 6.46(24)$ GeV^4	$\bar{h}_7 = -14.86(44)$ GeV^6	$h_1 = -0.17(2)$ GeV^2
LMD+V, $h_1 \neq 0$	B1	18/35	$\bar{h}_5 = 6.44(22)$ GeV^4	$\bar{h}_7 = -14.92(21)$ GeV^6	$h_1 = -0.17(2)$ GeV^2
LMD+V, $h_1 \neq 0$	B2	19/43	$\bar{h}_5 = 6.47(21)$ GeV^4	$\bar{h}_7 = -14.84(7)$ GeV^6	$h_1 = -0.17(2)$ GeV^2

Main improvement in normalization parameter, F_π for VMD and $\bar{h}_7$ for LMD+V. But more data also better determine the other parameters M_V or $\bar{h}_5$.

Results for $a_\mu^{\mathrm{LbyL};\pi^0}$

Model	Data	$a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{11}$
VMD	A0	$(57.2 \pm 4.0)_{JN}$
VMD	A1	$(57.7 \pm 2.1)_{JN}$
VMD	A2	$(57.3 \pm 1.1)_{JN}$
LMD+V, $h_1 = 0$	A0	$(72.3 \pm 3.5)^{*}_{JN}$
		$(79.8 \pm 4.2)_{MV}$
LMD+V, $h_1 = 0$	A1	$(73.0 \pm 1.7)^{*}_{JN}$
		$(80.5 \pm 2.0)_{MV}$
LMD+V, $h_1 = 0$	A2	$(72.5 \pm 0.8)^{*}_{JN}$
		$(80.0 \pm 0.8)_{MV}$
LMD+V, $h_1 \neq 0$	A0	$(72.4 \pm 3.8)^{*}_{JN}$
LMD+V, $h_1 \neq 0$	A1	$(72.9 \pm 2.1)^{*}_{JN}$
LMD+V, $h_1 \neq 0$	A2	$(72.4 \pm 1.5)^{*}_{JN}$
LMD+V, $h_1 \neq 0$	B0	$(71.9 \pm 3.4)^{*}_{JN}$
LMD+V, $h_1 \neq 0$	B1	$(72.4 \pm 1.6)^{*}_{JN}$
LMD+V, $h_1 \neq 0$	B2	$(71.8 \pm 0.7)^{*}_{JN}$

* error does not include uncertainty due to off-shellness of pion

Error in $a_\mu^{\mathrm{LbyL};\pi^0}$ related to model parameters determined by $\Gamma_{\pi^0\to\gamma\gamma}$ (normalization; not taken into account before) and $F(Q^2)$ is reduced as follows:

- Sets A0, B0: $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx 4 \times 10^{-11}$
- Sets A1, B1: $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx 2 \times 10^{-11}$ $(+\ \Gamma^{\mathrm{PrimEx}}_{\pi^0\to\gamma\gamma})$
- Sets A2, B2: $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx (0.7 - 1.1) \times 10^{-11}$ (+ KLOE-2 data)

VMD versus LMD+V with $h_1 = 0$

- Both VMD and LMD+V with $h_1 = 0$ can fit the data sets A0, A1 and A2 very well with essentially the same $\chi^2/d.o.f.$
- Nevertheless, the results for $a_\mu^{\mathrm{LbyL};\pi^0}$ differ by about 20%:

  $a_{\mu;\mathrm{VMD}}^{\mathrm{LbyL};\pi^0} \approx 57.5 \times 10^{-11}$

  $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} \approx 72.5 \times 10^{-11}$ (JN)

  $[a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} \approx 80 \times 10^{-11}$ (MV)]

- Due to the different behavior in these models of the fully off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ on all momentum variables.
- VMD model is known to have a wrong high-energy behavior $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, Q^2, Q^2) \sim 1/Q^4$ instead of $1/Q^2$ according to the OPE.
- The small final error of $\pm 1.1 \times 10^{-11}$ for the VMD model with only two parameters, F_π and M_V, which are both fixed by the width and form factor measurements, might therefore be very deceptive.

Conclusions

- Planned measurements at KLOE-2 can help to reduce some of the uncertainty in the (presumably !) numerically dominant pion exchange contribution to had. LbyL scattering.

- Error in $a_\mu^{\mathrm{LbyL};\pi^0}$ related to the model parameters determined by $\Gamma_{\pi^0\to\gamma\gamma}$ and $F(Q^2)$ will be reduced as follows:
  - $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx 4 \times 10^{-11}$ (with current data for $F(Q^2)$ + $\Gamma^{\mathrm{PDG}}_{\pi^0\to\gamma\gamma}$)
  - $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx 2 \times 10^{-11}$ (+ $\Gamma^{\mathrm{PrimEx}}_{\pi^0\to\gamma\gamma}$)
  - $\delta a_\mu^{\mathrm{LbyL};\pi^0} \approx (0.7 - 1.1) \times 10^{-11}$ (+ KLOE-2 data)

- Note that this error does not account for other potential uncertainties in $a_\mu^{\mathrm{LbyL};\pi^0}$, e.g. related to the off-shellness of the pion or the choice of model.

- Recall (in units of 10^{-11}):

  $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}(\mathrm{N,JN}) = 72 \pm 12$

  $\delta a_\mu^{\mathrm{LbyL}}(\mathrm{N,JN}) = 40; \quad \delta a_\mu^{\mathrm{LbyL}}(\mathrm{PdRV}) = 26; \quad \delta a_\mu^{\mathrm{LbyL}}(\mathrm{GFW}) = 91$

  $\delta a_\mu(\text{had. VP}) \approx 45; \quad \delta a_\mu(\text{exp [BNL]}) = 63; \quad \delta a_\mu(\text{future exp}) = 15$

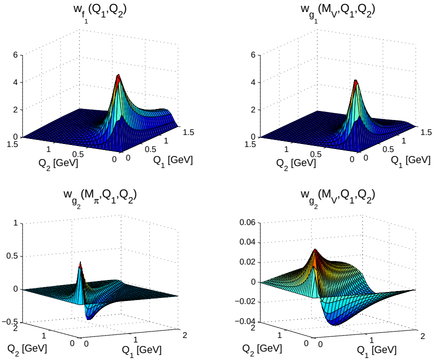
Backup slides

Relevant momentum regions in $a_\mu^{\mathrm{LbyL};\pi^0}$

In Knecht, Nyffeler '02, a 2-dimensional integral representation was derived for a certain class (VMD-like) of on-shell form factors (schematically):

$$a_\mu^{\mathrm{LbyL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2)\, f_i(Q_1, Q_2)$$

with universal weight functions w_i. Dependence on form factors resides in the f_i. Expressions with on-shell form factors are not valid as they stand. One needs to set form factor at external vertex to a constant to obtain pion-pole contribution. Expressions are valid for WZW and off-shell VMD form factors.



- $w_{f_1}(Q_1, Q_2)$ enters for WZW form factor. Tail leads to $\ln^2 \Lambda$ divergence for momentum cutoff Λ.
- $w_{g_1}(M_V, Q_1, Q_2)$ enters for VMD form factor.
- Relevant momentum regions are therefore around $0.25 - 1.25$ GeV. As long as form factors in different models lead to damping, we expect comparable results for $a_\mu^{\mathrm{LbyL};\pi^0}$, at the level of 20%. Similarly for η, η'.

Jegerlehner, Nyffeler '09 derived 3-dimensional integral representation for general form factors. Integration over $Q_1^2, Q_2^2, \cos\theta$, where $Q_1 \cdot Q_2 = |Q_1||Q_2|\cos\theta$.

Relevant momentum regions in $a_\mu^{\mathrm{LbyL;PS}}$

Result for pseudoscalar exchange contribution $a_\mu^{\mathrm{LbyL;PS}} \times 10^{11}$ for off-shell LMD+V and VMD form factors obtained with momentum cutoff Λ in 3-dimensional integral representation of JN '09 (integration over square). In brackets, relative contribution of the total obtained with $\Lambda = 20$ GeV.

Λ [GeV]	π^0 LMD+V ($h_3 = 0$)	π^0 LMD+V ($h_4 = 0$)	π^0 VMD	η VMD	η' VMD
0.25	14.8 (20.6%)	14.8 (20.3%)	14.4 (25.2%)	1.76 (12.1%)	0.99 (7.9%)
0.5	38.6 (53.8%)	38.8 (53.2%)	36.6 (64.2%)	6.90 (47.5%)	4.52 (36.1%)
0.75	51.9 (72.2%)	52.2 (71.7%)	47.7 (83.8%)	10.7 (73.4%)	7.83 (62.5%)
1.0	58.7 (81.7%)	59.2 (81.4%)	52.6 (92.3%)	12.6 (86.6%)	9.90 (79.1%)
1.5	64.9 (90.2%)	65.6 (90.1%)	55.8 (97.8%)	14.0 (96.1%)	11.7 (93.2%)
2.0	67.5 (93.9%)	68.3 (93.8%)	56.5 (99.2%)	14.3 (98.6%)	12.2 (97.4%)
5.0	71.0 (98.8%)	71.9 (98.8%)	56.9 (99.9%)	14.5 (99.9%)	12.5 (99.9%)
20.0	71.9 (100%)	72.8 (100%)	57.0 (100%)	14.5 (100%)	12.5 (100%)

π^0:

- Although weight functions plotted earlier are not applicable to off-shell LMD+V form factor, region below $\Lambda = 1$ GeV gives the bulk of the result: 82% for LMD+V, 92% for VMD.
- No damping from off-shell LMD+V form factor at external vertex since $\chi \neq 0$ (new short-distance constraint). Note: VMD falls off too fast, compared to OPE.

η, η':

- Mass of intermediate pseudoscalar is higher than pion mass $\rightarrow$ expect a stronger suppression from propagator.
- Peak of relevant weight functions shifted to higher values of Q_i. For η', vector meson mass is also higher $M_V = 859$ MeV. Saturation effect and the suppression from the VMD form factor only fully set in around $\Lambda = 1.5$ GeV: 96% of total for η, 93% for η'.

Slope of the transition form factor at the origin

- An important quantity is the slope of the form factor at the origin:

$$a \equiv m_\pi^2 \frac{1}{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)} \left. \frac{d\,\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q^2,0)}{d\,q^2} \right|_{q^2=0}$$

- Within ChPT, the slope is related to low-energy constants of the chiral Lagrangian of order p^6 in the odd intrinsic-parity sector. A precise measurement could help to distinguish between estimates of the low-energy constants, which have been made using various models: e.g. resonance Lagrangians (VMD, LMD, LMD+V), constituent quark models, holographic models (AdS/QCD), ...
- For time-like photon virtualities ($q^2 > 0$), the slope can be measured directly in the rare decay $\pi^0 \to e^+e^-\gamma$, but the current experimental uncertainty is big.
- The PDG quotes since many years $a = 0.032 \pm 0.004$.
- This value is essentially the result obtained by the CELLO collaboration for space-like momenta $q^2 = -Q^2 < 0$. CELLO fitted their data, collected for $Q^2 \geq 0.5$ GeV^2, with a simple VMD parametrization for the form factor and then used the analytical expression to obtain the slope.
- The potential model dependence of this extrapolation from rather large values of Q^2 to the origin is not accounted for by the PDG in the central value and the error for the slope parameter.

  Also contributions from loops in ChPT at order p^6, $a^{\mathrm{loops}}(\mu = M_\rho) = 0.005$ (Bijnens et al. '90), are not taken into account.